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AN ERGODIC THEOREM FOR FRÉCHET-VALUED RANDOM VARIABLES

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ABSTRACT. We generalize the classical ergodic theorem from real-valued random variables to Fréchet space-valued random variables and obtain this generalization as a direct corollary of the classical theorem. As an application we obtain several strong laws of large numbers for Fréchet-valued random variables. In a similar way we obtain a martingale theorem for Fréchet-valued random variables.

1. Preliminaries. Let F be a Fréchet space and q_k, $k \in \mathbf{N}$, be a family of seminorms generating the topology of F. Let $(\Omega, \mathcal{A}, P)$ be a probability space.

We shall consider the concept of integration according to Schäfke [4], [5], [6]. Let $\mathcal{F} = \{\sum_{\nu=1}^{n} x_\nu 1_{A_\nu}: x_\nu \in F, A_\nu \in \mathcal{A}\}$ by the system of F-valued simple functions. The integral of a simple function $X = \sum_{\nu=1}^{n} x_\nu 1_{A_\nu}$ is defined by $E(X) = \sum_{\nu=1}^{n} P(A_\nu) x_\nu$.

We can define the integral norm $\| \; \|: [0, \infty]^\Omega \to [0, \infty]$ (in Schäfke's notation $\| \; \|_3$) according to Theorem 5.6.1 of [5]. In our special case of a probability measure this integral norm is given by

$$\|g\| = \inf\{E(h): g \leqslant h, h \ \mathcal{A}\text{-measurable}\},$$

where $E(h)$ is the classical integral of an $\mathcal{A}$-measurable function h. By

$$\rho^*(X, Y) = \sum_{k=1}^{\infty} \frac{1}{2^k} \frac{\|q_k(X - Y)\|}{1 + \|q_k(X - Y)\|}$$

there is defined a pseudometric in the space of all functions from Ω to F. Then a function $X: \Omega \to F$ is P-integrable if there exists a sequence of simple functions $X_n: \Omega \to F$ such that $\rho^*(X_n, X) \to_{n \to \infty} 0$. Then $E(X_n)$, $n \in \mathbf{N}$, is Cauchy-convergent in F and $E(X) = \lim_{n\to\infty} E(X_n)$ is the P-integral of an P-integrable function X.

Let $L_1(\Omega, \mathcal{A}, P, F)$ be the system of all F-valued P-integrable functions. According to Theorem 2.4.5 of [5] we have $X \in L_1(\Omega, \mathcal{A}, P, F)$ iff there exists a sequence $X_n \in \mathcal{F}$, $n \in \mathbf{N}$ such that

(1) $X_n \to X$ P-a.e.

(2) $E(q_k(X_n - X_m)) \to_{n,m\to\infty} 0$ for all k.

Hence the range of an P-integrable function is contained P-a.e. in a separable Fréchet space. Let $\mathcal{A}_0$ be a sub-σ-field of $\mathcal{A}$ and $X \in L_1(\Omega, \mathcal{A}, P, F)$. Using

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the same techniques as Neveu [9, p. 100] for functions with values in a separable Banach space, one can define the conditional expectation $E(X|\mathcal{Q}_0)$.

2. The results. We show that the classical ergodic theorem holds true also for Fréchet-valued integrable functions. For the proof of this theorem we only use the classical ergodic theory for real-valued functions. We can immediately derive some strong laws of large numbers for Banach-valued and Fréchet-valued random variables which have been proved before by heavy techniques. With the same technique we show that a martingale theorem for Fréchet-valued random variables holds true. Compare with the heavy techniques used in [1, Satz 22.2] for Banach-valued variables.

THEOREM 1. *Let T be a measure preserving transformation of the probability space $(\Omega, \mathcal{Q}, P)$. Let F be a Fréchet space and X: $\Omega \to F$ be a P-integrable function. Then*

$$\frac{1}{n}\sum_{k=0}^{n-1} X \circ T^k \to E(X|\mathfrak{I}_T) \quad P\text{-a.e.}$$

where $\mathfrak{I}_T = \{A \in \mathcal{Q}: T^{-1}(A) = A\}$ is the σ-algebra of T-invariant sets.

PROOF. According to the classical ergodic theorem the assertion is true for all characteristic functions $X = 1_A$ with $A \in \mathcal{Q}$ and hence for the system $\mathfrak{F}$ of all simple functions. Now let $X \in L_1(\Omega, \mathcal{Q}, P, F)$. Then there exist $X_j \in \mathfrak{F}, j \in \mathbf{N}$, such that

$$\rho^*(X_j, X) \underset{j\to\infty}{\to} 0 \tag{1}$$

and

$$E(X_j|\mathfrak{I}_T) \underset{j\to\infty}{\to} E(X|\mathfrak{I}_T) \quad P\text{-a.e.} \tag{2}$$

We have for all $j, k \in \mathbf{N}$:

$$\begin{aligned} &q_k\left(\frac{1}{n}\sum_{\nu=0}^{n-1} X(T^\nu(\omega)) - E(X|\mathfrak{I}_T)\right) \\ &\quad < q_k\left(\frac{1}{n}\sum_{\nu=0}^{n-1} (X - X_j)(T^\nu(\omega))\right) + q_k\left(\frac{1}{n}\sum_{\nu=0}^{n-1} X_j(T^\nu(\omega)) - E(X_j|\mathfrak{I}_T)\right) \\ &\qquad + q_k\big(E(X_j|\mathfrak{I}_T) - E(X|\mathfrak{I}_T)\big). \end{aligned}$$

Since the assertion of the theorem holds for all X_j we obtain for all $j, k \in \mathbf{N}$:

$$\begin{aligned} &\overline{\lim_{n\to\infty}}\, q_k\left(\frac{1}{n}\sum_{\nu=0}^{n-1} X(T^\nu(\omega)) - E(X|\mathfrak{I}_T)\right) \\ &\quad < \overline{\lim_{n\to\infty}}\, \frac{1}{n}\sum_{\nu=0}^{n-1} q_k\big((X - X_j)(T^\nu(\omega))\big) + q_k\big(E(X_j|\mathfrak{I}_T) - E(X|\mathfrak{I}_T)\big). \end{aligned} \tag{3}$$

Let

$$f_j^{(k)}(\omega) = q_k\big((X - X_j)(\omega)\big)$$

and

$$g_j^{(k)}(\omega) = \overline{\lim_{n\to\infty}}\ \frac{1}{n} \sum_{\nu=0}^{n-1} f_j^{(k)}(T^\nu(\omega)).$$

By the classical ergodic theorem

$$g_j^{(k)} = E\big(f_j^{(k)} | \mathfrak{T}_T\big) \quad P\text{-a.e.}$$

and

$$E\big(E\big(f_j^{(k)} | \mathfrak{T}_T\big)\big) = E\big(f_j^{(k)}\big) \to 0$$

as $j \to \infty$ by (1). Therefore

$$g_j^{(k)} = E\big(f_j^{(k)} | \mathfrak{T}_T\big) \to 0 \quad P\text{-a.e.}$$

for a subsequence $j \in \mathbf{N}_0$. Now the assertion follows from (2) and (3).

THEOREM 2. *Let* $(\Omega, \mathcal{Q}, P)$ *be a probability space and* $\mathcal{Q}_n$, $n \in \mathbf{N}$, *be a sequence of sub-σ-fields of* $\mathcal{Q}$ *decreasing or increasing to the σ-field* $\mathcal{Q}_\infty$. *Let* F *be a Fréchet space and* $X\colon \Omega \to F$ *be a P-integrable function. Then*

$$E(X | \mathcal{Q}_n) \to E(X | \mathcal{Q}_\infty) \quad P\text{-a.e.}$$

PROOF. According to the classical martingale theorem the theorem is true for all characteristic functions $X = 1_A$ with $A \in \mathcal{Q}$ and hence for the system $\mathfrak{F}$ of all simple functions. Now let $X \in L_1(\Omega, \mathcal{Q}, P, F)$. Then there exist $X_j \in \mathfrak{F}$, $j \in \mathbf{N}$, such that

$$\rho^*(X_j, X) \underset{j\to\infty}{\to} 0. \tag{1}$$

We have for all $j, k, n \in \mathbf{N}$

$$\begin{aligned} q_k(E(X|\mathcal{Q}_n) - E(X|\mathcal{Q}_\infty)) &\leq q_k\big(E(X - X_j|\mathcal{Q}_n)\big) + q_k\big(E(X_j|\mathcal{Q}_n) - E(X_j|\mathcal{Q}_\infty)\big) \\ &\quad + q_k\big(E(X_j - X|\mathcal{Q}_\infty)\big) \\ &\leq E\big(q_k(X - X_j)|\mathcal{Q}_n\big) + q_k\big(E(X_j|\mathcal{Q}_n) \\ &\quad - E(X_j|\mathcal{Q}_\infty)\big) + E\big(q_k(X_j - X)|\mathcal{Q}_\infty\big). \end{aligned}$$

Since the assertion of the theorem holds for all X_j, we obtain for all $j, k \in \mathbf{N}$:

$$\begin{aligned} &\overline{\lim_{n\to\infty}}\ q_k(E(X|\mathcal{Q}_n) - E(X|\mathcal{Q}_\infty)) \\ &\qquad \leq \overline{\lim_{n\to\infty}}\ E\big(q_k(X - X_j)|\mathcal{Q}_n\big) + E\big(q_k(X_j - X)|\mathcal{Q}_\infty\big). \end{aligned}$$

Using the classical martingale theorem we obtain for all $j, k \in \mathbf{N}$:

$$\overline{\lim_{n\to\infty}}\ q_k(E(X|\mathcal{Q}_n) - E(X|\mathcal{Q}_\infty)) \leq 2E\big(q_k(X_j - X)|\mathcal{Q}_\infty\big). \tag{2}$$

By (1) for all $k \in \mathbf{N}$:

$$E\left(E\left(q_k(X_j - X)|\mathcal{A}_\infty\right)\right) = E\left(q_k(X_j - X)\right) \to 0$$

as $j \to \infty$. Therefore for each $k \in \mathbf{N}$

$$E\left(q_k(X_j - X)|\mathcal{A}_\infty\right) \to 0 \quad P\text{-a.e.}$$

for a subsequence $j \in \mathbf{N}_1$. Hence the assertion follows from (2).

It is also possible to obtain in this direct way convergence results for convergence in the pth mean for Theorems 1 and 2.

Considering the canonical process associated to a stochastic process one obtains from Theorem 1 as in the classical case:

COROLLARY 3. *Let X_n, $n \in \mathbf{N}$, be a stationary process defined on a probability space $(\Omega, \mathcal{A}, P)$ with values in a Fréchet space F. Let X_1 be P-integrable, then*

$$\frac{1}{n} \sum_{k=1}^{n} X_k \underset{n\to\infty}{\to} E\left(X_1|\mathfrak{I}(X_n: n \in \mathbf{N})\right) \quad P\text{-a.e.}$$

where $\mathfrak{I}(X_n: n \in \mathbf{N})$ is the system of all invariant sets of the process X_n, $n \in \mathbf{N}$.

Since $\mathfrak{I}(X_n: n \in \mathbf{N})$ is contained in the σ-field of terminal sets of the process X_n, $n \in \mathbf{N}$, the zero-one law of Kolmogorov implies that every invariant set has measure 0 or 1, if X_n, $n \in \mathbf{N}$, are independent. Hence we obtain

COROLLARY 4. *Let X_n, $n \in \mathbf{N}$, be independent and identically distributed random variables with values in a Fréchet space F. Let X_1 be P-integrable then*

$$\frac{1}{n} \sum_{k=1}^{n} X_k \underset{n\to\infty}{\to} E(X_1) \quad P\text{-a.e.}$$

Using the fact that a measurable random variable X, with values in a separable Fréchet space, fulfilling $E[q_k(X)] < \infty$ for each $k \in \mathbf{N}$, belongs to $L_1(\Omega, \mathcal{A}, P, F)$, we obtain the theorem of Taylor and Padgett [7]. Hence we obtain Theorems 4.1.1 and 6.1.2 of [3] as special cases. We can also generalize Theorem 4.3.1 of [3] from Banach spaces with a separable dual to Fréchet spaces with a separable dual.

COROLLARY 5. *Let X_n, $n \in \mathbf{N}$, be a stationary process defined on a probability space with values in a Fréchet space with a separable dual space. Let $E(X_1) = 0$, $E(q_k^2(X_1)) < \infty$ for all $k \in \mathbf{N}$. Let the process be weakly orthogonal, i.e. $E(f(X_n)f(X_m)) = 0$ for all $n \neq m$ and all continuous linear functionals of F, then*

$$\frac{1}{n} \sum_{k=1}^{n} X_k \to 0 \quad P\text{-a.e.}$$

PROOF. According to Corollary 2 we obtain

$$\frac{1}{n} \sum_{k=1}^{n} X_k \underset{n\to\infty}{\to} E\left(X_1|\mathfrak{I}(X_n: n \in \mathbf{N})\right). \tag{1}$$

Now using the same method as in [**3**, p. 56] we obtain the assertion.

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