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POINCARÉ SERIES OF MODULES OVER LOCAL RINGS

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ABSTRACT. There is a well known conjecture, originally made by Kaplansky, that the Poincaré series $P_R^M(z)$ of a finitely generated module M over a local ring R is a rational function. Two reductions of this conjecture are made, one to the case where M is artinian (for a fixed R) and the other to the case where R is artinian.

Introduction. Let R be a local ring with maximal ideal $\mathbf{m}$ and residue field $k = R/\mathbf{m}$. All R-modules considered are assumed to be finitely generated and all unspecified tensor products are over R.

For any R-module M, we may consider the formal power series

$$P_R^M(z) = \sum_{i=0}^{\infty} B_i z^i$$

where $B_i = \dim_k \operatorname{Tor}_i^R(M, k)$. There is a well-known conjecture, originally made by Kaplansky, that $P_R^M(z)$ is a rational function for all local rings R and all R-modules M. We make two reductions of this conjecture.

THEOREM 1. *The series P_R^M is rational for all local rings R and R-modules M if and only if P_R^M is rational for all artinian local rings R and R-modules M.*

THEOREM 2. *For any local ring R, P_R^M is rational for all R-modules M if and only if P_R^M is rational for all artinian R-modules M.*

The proofs will follow after some lemmas. Theorem 1 for the case $M = k$ first appeared in [3]. However, the proof included here is much shorter. In [2], Gulliksen and Ghione note that the author's result together with a result of Gulliksen's imply that P_R^M is rational for all local rings R and R-modules M if and only if P_R^k is rational for all artinian local rings R. Theorem 2 is new.

LEMMA 1. *Let K be the Koszul complex of R and M an R-module. Then there is an integer n_0 such that for all $n \geqslant n_0$, the induced homomorphism*

$$H(\mathbf{m}^{n+1}M \otimes K) \to H(\mathbf{m}^n M \otimes K)$$

is zero.

PROOF. Since

$$H(\mathbf{m}^n M \otimes K) = \frac{(\mathbf{m}^n M \otimes K) \cap Z(M \otimes K)}{\mathbf{m}^n B(M \otimes K)}$$

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and since $H(\mathbf{m}^n M \otimes K)$ is a vector space over k,

$$\mathbf{m}((\mathbf{m}^n M \otimes K) \cap Z(M \otimes K)) \subset \mathbf{m}^n B(M \otimes K).$$

Then by the Artin-Rees lemma, there is an integer n_0 such that for $n \geqslant n_0$,

$$\begin{aligned}\mathbf{m}^{n+1}(M \otimes K) \cap Z(M \otimes K) &\subset \mathbf{m}(\mathbf{m}^n(M \otimes K) \cap Z(M \otimes K)) \\ &\subset \mathbf{m}^n B(M \otimes K).\end{aligned}$$

Thus the induced homomorphism

$$H(\mathbf{m}^{n+1}M \otimes K) \to H(\mathbf{m}^n M \otimes K)$$

is zero.

LEMMA 2. *Let A be a differential graded R-algebra and E a differential graded A-module which is free as a graded A-module (i.e. forgetting differentials) and such that $dE \subset \mathbf{m}E$. Let M and N be R-modules such that $\mathbf{m}M \subset N \subset M$, and such that the canonical map g: $N \otimes E \to M \otimes E$ is injective. Then if the induced homomorphism $H(N \otimes A) \to H(M \otimes A)$ is zero, so is the induced homomorphism $H(N \otimes E) \to H(M \otimes E)$.*

PROOF. Assume that $H(N \otimes A) \to H(M \otimes A)$ is zero.
Then

$$f(Z(N \otimes A)) \subset B(M \otimes A), \tag{1}$$

where f is the canonical map f: $N \otimes A \to M \otimes A$. Define a filtration on E as follows:

$$F_p E = \sum_{i \geqslant p} A_i E.$$

We first show that

$$g\big(Z(N \otimes E) \cap (N \otimes F_p E)\big) \subset B_p(M \otimes A)E + g(N \otimes F_{p+1}E). \tag{2}$$

Let $S_1, S_2, \ldots$ be the elements of degree $n - p$ of a homogeneous A-basis for E. Then if $z \in Z(N \otimes E) \cap (N \otimes F_p E)$, there are $a_i \in N \otimes A_p$ such that $z - \Sigma a_i S_i \in N \otimes F_{p+1}E$. It follows that $\Sigma(da_i)S_i \in N \otimes F_p E$ so each $da_i = 0$. But then because of (1), each $f(a_i) \in B_p(M \otimes A)$ and since $g(\Sigma a_i S_i) = \Sigma f(a_i) S_i$, (2) is proved.

Next, observe that the derivation formula

$$d(xy) = (dx)y + (-1)^{\deg x} x(dy)$$

shows that

$$\big(d(M \otimes A_{p+1})\big)E \subset d\big((M \otimes A_{p+1})E\big) + (M \otimes A_{p+1})dE$$

and since $dE \subset \mathbf{m}E$ and $\mathbf{m}M \subset N$,

$$\big(d(M \otimes A_{p+1})\big)E \subset d(M \otimes E) + g(N \otimes F_{p+1}E). \tag{3}$$

Now by (2) and (3) and the injectivity of g,

$$g\big(Z(N\otimes E)\cap(N\otimes F_pE)\big)$$
$$\subset g\big(Z(N\otimes E)\cap(N\otimes F_{p+1}E)\big)+B(M\otimes E).$$

Since E is A-free, $F_0E=E$ and also for any n, $F_pE\cap E_n=0$ for $p>n$. Hence $g(Z(N\otimes E))\subset B(M\otimes E)$ and $H(N\otimes E)\to H(M\otimes E)$ is zero.

PROOF OF THEOREM 2. By Lemma 1, there exists an integer n_0 such that for $n\geqslant n_0$

$$H(\mathbf{m}^{n+1}M\otimes K)\to H(\mathbf{m}^nM\otimes K)$$

is zero. Let X be a minimal free resolution of k. By [1], X is free as a graded K-module. Thus Lemma 2 applies showing that

$$H(\mathbf{m}^{n+1}M\otimes X)\to H(\mathbf{m}^nM\otimes X) \tag{4}$$

is zero for $n\geqslant n_0$. It also follows that

$$H(\mathbf{m}^{n+1}M\otimes X)\to H(M\otimes X)$$

is zero for $n\geqslant n_0$.

Thus the short exact sequences

$$0\to\mathbf{m}^nM\to M\to M/\mathbf{m}^nM\to 0$$

induce short exact sequences

$$0\to\operatorname{Tor}_p^R(M,k)\to\operatorname{Tor}_p^R(M/\mathbf{m}^nM,k)\to\operatorname{Tor}_{p-1}^R(\mathbf{m}^nM,k)\to 0$$

for $n\geqslant n_0+1$, showing that

$$P_R^{M/\mathbf{m}^nM}=P_R^M+zP_R^{\mathbf{m}^nM}. \tag{5}$$

We now compute $P_R^{\mathbf{m}^nM}$ for $n\geqslant n_0+1$. Because of (4), the short exact sequences

$$0\to\mathbf{m}^{n+i+1}M\to\mathbf{m}^{n+i}M\to\mathbf{m}^{n+i}M/\mathbf{m}^{n+i+1}M\to 0$$

induce short exact sequences

$$0\to\operatorname{Tor}_p^R(\mathbf{m}^{n+i}M,k)\to\operatorname{Tor}_p^R(k,k)\otimes(\mathbf{m}^{n+i}M/\mathbf{m}^{n+i+1}M)$$
$$\to\operatorname{Tor}_{p-1}^R(\mathbf{m}^{n+i+1}M,k)\to 0$$

and thus

$$c_{n+i}P_R^k=P_R^{\mathbf{m}^{n+i}M}+zP_R^{\mathbf{m}^{n+i+1}M}$$

for all $i\geqslant 0$ where

$$c_r=\dim_k\mathbf{m}^rM/\mathbf{m}^{r+1}M.$$

Let $H(z)=\sum_{i=0}^\infty c_{n+i}z^i$. By the Hilbert theory, c_{n+i} is a polynomial in i for large i so $H(z)$ is a rational function. Now

$$P_R^{\mathbf{m}^nM}=\sum_{i=0}^\infty(-1)^iz^ic_{n+i}P_R^k=H(-z)P_R^k. \tag{6}$$

Then (5) and (6) show that for $n\geqslant n_0+1$, P_R^M is rational if both $P_R^{M/\mathbf{m}^nM}$ and P_R^k are rational. Since $M/\mathbf{m}^nM$ and k are artinian modules the theorem follows.

PROOF OF THEOREM 1. Assume that all modules over artinian local rings have rational Poincaré series and let R be any local ring and M an R-module. By Theorem 2, it is enough to prove that P_R^M is rational for artinian R-modules M. So assume that $\mathbf{m}^r M = 0$.

As in the proof of Theorem 1, there is an integer n_0 such that for $n \geqslant n_0$

$$\operatorname{Tor}^R(\mathbf{m}^{n+1}, k) \to \operatorname{Tor}^R(\mathbf{m}^n, k) \tag{7}$$

is zero. Let $n \geqslant \max(r, n_0)$.

Then from (6)

$$P_R^{\mathbf{m}^{n+1}} = H(-z)P_R^k$$

where $H(z) = \sum_{i=0}^{\infty} \dim(\mathbf{m}^{n+i+1}/\mathbf{m}^{n+i+2})z^i$. So

$$P_R^{R/\mathbf{m}^{n+1}} = 1 + zH(-z)P_R^k.$$

We will show that for any R-module N such that $\mathbf{m}^n N = 0$,

$$P_{R/\mathbf{m}^{n+1}}^N = \frac{P_R^N}{1 - z\left(P_R^{R/\mathbf{m}^{n+1}} - 1\right)}. \tag{8}$$

With $N = k$, this shows that P_R^k is rational and thus that $P_R^{R/\mathbf{m}^{n+1}}$ is rational. Then with $N = M$, it follows that P_R^M is rational.

To prove (8), let X be a minimal algebra resolution [1] of k over R. We put $R^* = R/\mathbf{m}^{n+1}$, $m^* = \mathbf{m}/\mathbf{m}^{n+1}$, and $X^* = X \otimes R^*$. For each $i \geqslant 2$, let F_i be a free R^*-module such that

$$F_i \otimes k \cong H_{i-1}(X^*).$$

Because of (7) $H(\mathbf{m}^{n+1}X) \to H(\mathbf{m}^n X)$ is zero so $H(\mathbf{m}^n X^*) \to H(X^*)$ is surjective. Thus we can choose R-homomorphism η: $F_i \to \mathbf{m}^n X_{i-1}^*$ such that the diagrams

$$\begin{array}{ccc} F_i & \to & \mathbf{m}^n X_{i-1}^* \\ \downarrow & & \downarrow \\ F_i \otimes k & \approx & H_{i-1}(X^*) \end{array}$$

commute. Now define a complex $Y = X^* \otimes T(F)$ with differential given by

$$\begin{aligned} d(x \otimes f_1 \otimes \cdots \otimes f_r) &= dx \otimes f_1 \otimes \cdots \otimes f_r \\ &\quad + (-1)^{\deg x} x\eta(f_1) \otimes f_2 \otimes \cdots \otimes f_r \end{aligned}$$

for $x \in X^*$ and $f_i \in F$.

It is easily seen that $d^2 = 0$ and that

$$d(y \otimes f) = dy \otimes f \pmod{X^*} \tag{9}$$

for $y \in Y$. Clearly $dY \subset \mathbf{m}^* Y$. The definition of η shows that

$$Z_i(X^*) \subset B_i(X^* \oplus F) \subset B_i(Y)$$

for $i > 0$. If $x + f \in Z_i(X^* \oplus F)$ then $\eta(f) \in B_{i-1}(X^*)$ so

$$f \in \mathbf{m}^* F_i \subset d(X_1^* \otimes F_i) + X_i^*.$$

Thus

$$Z_i(X^* \oplus F) \subset B_i(X^* \oplus F \oplus (X^* \otimes F)). \tag{10}$$

In particular, $H_1(Y) = H_2(Y) = 0$. Assume that $H_i(Y) = 0$ for $i < p$. By (9)

$$Z_p(Y) \subset X^* \oplus F \oplus \coprod_{i=1}^{p-2} (Z_i(Y) \otimes F_{p-i}) \subset X^* \oplus F \oplus \coprod_{i=1}^{p-2} d(Y_{i+1} \otimes F_{p-i})$$

by induction. So by (10) Y is acyclic and hence a minimal resolution of k over R^*. Since $\mathfrak{m}^n N = 0$, the differential in $N \otimes Y$ is given by $d(x \otimes f_1 \otimes \cdots \otimes f_n) = dx \otimes f_1 \otimes \cdots \otimes f_n$ for $x \in N \otimes X^*, f_i \in F$. Thus

$$H(N \otimes Y) \cong H(N \otimes X^*) \otimes T(F) \cong H(N \otimes X) \otimes T(F).$$

Formula (8) follows.

Corollary. *Let R be a local ring and M an R-module of finite length. Then there is an integer n_0 such that for $n \geqslant n_0$,*

$$P^M_{R/\mathfrak{m}^n} = P^M_R / \left(1 - z^2 H(-z) P^k_R\right)$$

where

$$H(z) = \sum_{i=0}^{\infty} \dim_k(\mathfrak{m}^{n+i}/\mathfrak{m}^{n+i+1}) z^i.$$

References

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