A CHARACTERIZATION OF C*-SUBALGEBRAS

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Abstract. Let A be a closed linear subspace of a C*-algebra B. Adjoin, if necessary, the identity 1 to B. Then A is a C*-subalgebra if and only if, for each x in A, the elements x* and |x| + 1 − | |x| − 1| are in A. If 1 is in A, then A is a C*-subalgebra if and only if |x| is in A for each x in A. Here |x| denotes the unique positive square root of x*x in B.

This note is entirely devoted to the proof of and a discussion on the following result.

Theorem. Let A be a closed linear subspace of a complex C*-algebra B. A is a C*-subalgebra of B if and only if, for each x in A, the elements x* and |x| + 1 − | |x| − 1| are in A.

Here |x| denotes the unique positive square root of x*x in B. The element |x| + 1 − | |x| − 1| is to be interpreted as g(|x|), where g(t) = t + 1 − |t − 1|, t ≥ 0. Since g(0) = 0, the continuous functional calculus shows that g(|x|) is in B; see Proposition 1.5.6, p. 11 of Dixmier [3]. In fact it follows that |x| + 1 − | |x| − 1| is in the C*-algebra generated by x*x.

Proof. The necessity is included in the above remark.

Sufficiency. Let x be in A. The equality

\[ |x|^2 = \int_0^{\infty} \left( (|x| + t - |x| - t) - (|x| + \frac{1}{2} t - |x| - \frac{1}{2} t) \right) \, dt \]

proves that x*x = |x|^2 is in A. The polarisation formula

\[ 4a*b = |a + b|^2 - |a - b|^2 + i(|a - ib|^2 - |a + ib|^2) \]

shows that a*b is in A whenever a and b are in A. Since x* is in A, whenever x is in A, we see that A is a C*-subalgebra.

Corollary 1. Let A and B be as in the Theorem. If each element of A is normal, then A is a C*-subalgebra of B if and only if, for each x in A, the element |x| + 1 − | |x| − 1| is in A.

Proof. The sufficiency part needs some explanation. Let x be in A and n in N. Since x*x = xx*, it follows from the necessity part of the Theorem and from the polar decomposition of x that

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\[ \|2x^* - x^*(n|x| + 1 - |n|x| - 1)| = \|(n|x| - 1 - n|x| + 1) x^*\| \]
\[ = \|(n|x| - 1 - n|x| + 1)|x|\| \]
\[ < \sup \{(nt - 1 - nt + 1) t; 0 < t < \|x\| \} \]
\[ < (2n)^{-1}. \]

So \( x^* \) is in \( A \) by the proof of the sufficiency part of the Theorem. □

**Corollary 2.** Let \( A \) and \( B \) be as in the Theorem. Assume that \( B \) has identity 1. If \( 1 \) is in \( A \), then \( A \) is a \( C^\ast \)-subalgebra if and only if \( |x| \) is in \( A \) for each \( x \) in \( A \).

**Proof.** The necessity is clear and the sufficiency follows from the equality
\[ 4x^* = |x + 1|^2 - |x - 1|^2 + i(|x - i|^2 - |x + i|^2), \quad x, y \in B. \]

**Corollary 3.** Let \( A \) be a closed subspace of a commutative \( C^\ast \)-algebra \( B \). \( A \) is a \( C^\ast \)-subalgebra if and only if with each \( x \) in \( A \) the element \( \min(1, |x|) \) is in \( A \) too.

Corollary 3 is an improvement of Proposition in Dellacherie [2, p. 52]. The following result is due to the referee. The result should be compared with R. V. Kadison [4, Theorem 6, p. 499].

**Corollary 4.** A continuous linear map \( \Lambda \) between two \( C^\ast \)-algebras \( B \) and \( B' \) is a \( C^\ast \)-algebra homomorphism if and only if
\[ \Lambda(|x| + 1 - |x| - 1) = |\Lambda(x)| + 1 - |\Lambda(x)| - 1 \]
for each \( x \) in \( B \).

**Corollary 5.** Let \( A \) be a \( C^\ast \)-subalgebra of \( B \). Assume that \( B \) has identity 1. Then \( A \) contains \( 1 \) if and only if, for some \( x \) in \( A \), \( x^{-1} \) exists in \( B \).

**Proof.** For the sufficiency part we notice that \( |x|^{-1} \) exists whenever \( x^{-1} \) exists. Hence let \( x \) in \( A \) be an element which is invertible in \( B \). Then there is \( \delta > 0 \) such that \( |x| > \delta \). Hence \( 2\delta = |x| + \delta - |x| - \delta | \) is in \( A \). □

**Corollary 6.** Let \( A \) be a vector space of real-valued functions. Assume that a real-valued function \( f \) is in \( A \) whenever there exists a sequence \( \{f_n; n \in \mathbb{N}\} \) in \( A \) such that
\[ \lim_{n \to \infty} \sup_{x \in \{|f| < m\}} |f(x) - f_n(x)| = 0 \]
for each positive \( m \). Then \( A \) is an algebra for the pointwise operations if and only if \( A \) is a Stone lattice.

Here a Stone lattice \( A \) is a vector space of functions for which \( \min(1, f) \) is in \( A \) for each \( f \) in \( A \); see H. Bauer [1, p. 194].

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REFERENCES


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