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A CHARACTERIZATION OF C^*-SUBALGEBRAS

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ABSTRACT. Let A be a closed linear subspace of a C^*-algebra B. Adjoin, if necessary, the identity 1 to B. Then A is a C^*-subalgebra if and only if, for each x in A, the elements x^* and $|x| + 1 - ||x| - 1|$ are in A. If 1 is in A, then A is a C^*-subalgebra if and only if $|x|$ is in A for each x in A. Here $|x|$ denotes the unique positive square root of x^*x in B.

This note is entirely devoted to the proof of and a discussion on the following result.

THEOREM. *Let A be a closed linear subspace of a complex C^*-algebra B. A is a C^*-subalgebra of B if and only if, for each x in A, the elements x^* and $|x| + 1 - ||x| - 1|$ are in A.*

Here $|x|$ denotes the unique positive square root of x^*x in B. The element $|x| + 1 - ||x| - 1|$ is to be interpreted as $g(|x|)$, where $g(t) = t + 1 - |t - 1|$, $t \geqslant 0$. Since $g(0) = 0$, the continuous functional calculus shows that $g(|x|)$ is in B; see Proposition 1.5.6, p. 11 of Dixmier [3]. In fact it follows that $|x| + 1 - ||x| - 1|$ is in the C^*-algebra generated by x^*x.

PROOF. The necessity is included in the above remark.

SUFFICIENCY. Let x be in A. The equality

$$|x|^2 = \int_0^\infty \left\{ (|x| + t - ||x| - t|) - \left(|x| + \tfrac{1}{2}t - \left||x| - \tfrac{1}{2}t\right|\right) \right\} dt$$

proves that $x^*x = |x|^2$ is in A. The polarisation formula

$$4a^*b = |a + b|^2 - |a - b|^2 + i(|a - ib|^2 - |a + ib|^2)$$

shows that a^*b is in A whenever a and b are in A. Since x^* is in A, whenever x is in A, we see that A is a C^*-subalgebra. □

COROLLARY 1. *Let A and B be as in the Theorem. If each element of A is normal, then A is a C^*-subalgebra of B if and only if, for each x in A, the element $|x| + 1 - ||x| - 1|$ is in A.*

PROOF. The sufficiency part needs some explanation. Let x be in A and n in $\mathbf{N}$. Since $x^*x = xx^*$, it follows from the necessity part of the Theorem and from the polar decomposition of x that

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$$\begin{aligned}\|2x^* - x^*(n|x| + 1 - |n|x| - 1|)\| &= \|(|n|x| - 1| - n|x| + 1)x^*\| \\ &= \|(|n|x| - 1| - n|x| + 1)|x|\| \\ &\leqslant \sup\{(|nt - 1| - nt + 1)t;\ 0 \leqslant t \leqslant \|x\|\} \\ &\leqslant (2n)^{-1}.\end{aligned}$$

So x^* is in A by the proof of the sufficiency part of the Theorem. □

COROLLARY 2. *Let A and B be as in the Theorem. Assume that B has identity* 1. *If* 1 *is in A, then A is a C^*-subalgebra if and only if $|x|$ is in A for each x in A.*

PROOF. The necessity is clear and the sufficiency follows from the equality

$$4x^* = |x + 1|^2 - |x - 1|^2 + i(|x - i|^2 - |x + i|^2), \qquad x, y \in B. \quad \square$$

COROLLARY 3. *Let A be a closed subspace of a commutative C^*-algebra B. A is a C^*-subalgebra if and only if with each x in A the element* $\min(1, |x|)$ *is in A too.*

Corollary 3 is an improvement of Proposition in Dellacherie [**2**, p. 52]. The following result is due to the referee. The result should be compared with R. V. Kadison [**4**, Theorem 6, p. 499].

COROLLARY 4. *A continuous linear map Λ between two C^*-algebras B and B' is a C^*-algebra homomorphism if and only if*

$$\Lambda(|x| + 1 - |\,|x| - 1|) = |\Lambda(x)| + 1 - |\,|\Lambda(x)| - 1|$$

for each x in B.

COROLLARY 5. *Let A be a C^*-subalgebra of B. Assume that B has identity* 1. *Then A contains* 1 *if and only if, for some x in A, x^{-1} exists in B.*

PROOF. For the sufficiency part we notice that $|x|^{-1}$ exists whenever x^{-1} exists. Hence let x in A be an element which is invertible in B. Then there is $\delta > 0$ such that $|x| \geqslant \delta$. Hence $2\delta = |x| + \delta - |\,|x| - \delta|$ is in A. □

COROLLARY 6. *Let A be a vector space of real-valued functions. Assume that a real-valued function f is in A whenever there exists a sequence $\{f_n;\ n \in \mathbf{N}\}$ in A such that*

$$\lim_{n\to\infty} \sup_{x \in \{|f| \leqslant m\}} |f(x) - f_n(x)| = 0$$

for each positive m. Then A is an algebra for the pointwise operations if and only if A is a Stone lattice.

Here a Stone lattice A is a vector space of functions for which $\min(1, f)$ is in A for each f in A; see H. Bauer [**1**, p. 194].

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References

1. H. Bauer, *Probability theory and elements of measure theory*, Holt, Rinehart and Winston, New York, 1972.

2. Cl. Dellacherie, *Un complément au théorème de Stone-Weierstrass*, Lecture Notes in Math., vol. 39, Springer-Verlag, Berlin, 1967, pp. 52–53. MR **37** #4584.

3. J. Dixmier, *Les C^*-algèbres et leurs représentations*, Gauthier-Villars, Paris, 1969. MR **39** #7442.

4. R.V. Kadison, *A generalized Schwarz inequality and algebraic invariants for operator algebras*, Ann. of Math. (2) **56** (1952), 494–503. MR **14**, 481.

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