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FINITE DIMENSIONAL PERTURBATIONS

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ABSTRACT. Let A be a normal operator on the Hilbert space $\mathcal{H}$ and let B be an operator of finite rank, rank $B = m$, such that $A + B$ is normal. Moreover let E (resp. F) denote the spectral projections of A (resp. $A + B$) for the set $\{\zeta \in \mathbf{C} \mid |\zeta - \lambda| < \alpha\}$. Then $\dim E - m < \dim F < \dim E + m$.

Weyl's theorem states that a perturbation of a selfadjoint operator A by a selfadjoint compact operator B leaves the essential spectrum $\sigma_{\text{ess}}(A)$ of A invariant. If B has finite rank one can say more.

THEOREM. *Let A be a normal, not necessarily bounded operator on the Hilbert space $\mathcal{H}$, and let E denote the spectral projection of A for the set $\Lambda = \{\zeta \in \mathbf{C} \mid |\zeta - \lambda| \leqslant \alpha\}$. Let B be an operator of finite rank,* rank $B = m$, *such that $A + B$ is normal. Then the spectral projection F of $A + B$ for Λ satisfies* $\dim E - m \leqslant \dim F \leqslant \dim E + m$, *where negative dimension for a projection F means $F = 0$.*

This theorem can be considered as a generalization of results of Wolf [**5**] and Hochstadt [**4**], who considered one dimensional perturbations of selfadjoint (compact) operators. Unlike [**4**] and [**5**] we do not use resolvent methods in the proof of the theorem.

LEMMA. *Let A be a normal operator and P a projection of dimension n whose range $\mathfrak{R}_P$ is contained in the domain $\mathfrak{D}_A$ of A. Assume $|AP| \leqslant \alpha$. Then the spectral projection Q of A for the set $\Lambda = \{z \in \mathbf{C} \mid |z| \leqslant \alpha\}$ satisfies* $\dim Q \geqslant n$.

PROOF. Assume $\dim Q < n$. By [**1**] or [**3**] we can write

$$\mathcal{H} = \mathcal{H}_0 \oplus (\mathcal{H}_1 + \mathcal{H}_1),$$

$$P = P_0 + \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = P_0 + P_1, \qquad Q = Q_0 + \begin{pmatrix} c^2 & cs \\ cs & s^2 \end{pmatrix} = Q_0 + Q_1, \quad (1)$$

where

(i) $\mathcal{H}_0$ reduces P and Q, $P_0 = P|\mathcal{H}_0$, $Q_0 = Q|\mathcal{H}_0$,

(ii) $P_0Q_0 = Q_0P_0$,

(iii) c and s are commuting selfadjoint operators on $\mathcal{H}_1$ with $c^2 + s^2 = 1$ and $0, 1 \notin \sigma(c^2)$, the spectrum of c^2.

Since $\dim P_1 = \dim Q_1$ we see $\dim P_0 > \dim Q_0$. Hence the projection

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$R = P_0 - P_0 \cdot Q_0$ is nontrivial. It satisfies $R \leqslant P$ and $RQ = 0$. Hence we have $\alpha \geqslant |AP| \geqslant |AR| = |A(1 - Q)R| > \alpha$, because the operator $A|(1 - Q)\mathcal{H}$ has its spectrum outside Λ. This contradiction proves the lemma.

PROOF OF THE THEOREM. Let R be the range-projection of B^*B. Then $\dim R = m$. By **[1]** or **[3]** we can write as in the proof of the lemma (1) $E = E_0 + E_1$, $R = R_0 + R_1$. Since $\dim E_1 = \dim R_1$ we see $\dim E - \dim R = \dim E_0 - \dim R_0 = k$. Without loss of generality we may assume k to be positive. Then the projection $P = E_0 - E_0 \cdot R_0$ satisfies $P \leqslant E$, $PR = 0$ and $\dim P \geqslant k$. Thus we get $\mathfrak{R}_P \in \mathfrak{D}_{A+B}$ and $|(A + B - \lambda)P| = |(A - \lambda)P| \leqslant \alpha$ because $BP = BRP = 0$. From the lemma we obtain now $\dim F \geqslant \dim E - m$. The other inequality follows from an application of the above result to $(A + B) - B$.

Since there is no restriction on the norm of the perturbation, it is easy to see with finite dimensional examples that the theorem is close to optimal.

The theorem and the lemma yield the following generalization of **[4]** and **[5]**.

COROLLARY 1. *Let A be a selfadjoint operator on the Hilbert space $\mathcal{H}$ and let $B = B_1 + B_2$ be a compact selfadjoint operator with rank $B_1 = n$ and $|B_2| = \beta$, $\beta \geqslant 0$. Let $\lambda_1 < \lambda_2 < \cdots < \lambda_l$ be successive eigenvalues of A of multiplicities n_i, $i = 1, \ldots, l$. Assume $[\lambda_1 - \beta, \lambda_l + \beta]$ does not intersect the continuous spectrum of A. Then there are at least $\sum_{i=1}^{l} n_i - n$ eigenvalues of $A + B$ in $[\lambda_1 - \beta, \lambda_l + \beta]$, where eigenvalues are counted according to their multiplcity.*

An easy consequence of this corollary is Weyl's theorem. For $\beta = 0$ the corollary also shows that a rank n perturbation can at most generate or destroy n eigenvalues in the connected components of $\mathbf{R} \setminus \sigma_{\text{ess}}(A)$.

For a simple proof of the results of Hochstadt **[4]** and Wolf **[5]** we shall restrict ourselves now to rank 1 perturbations.

Let A be a normal operator on $\mathcal{H}$ and let B be defined by $B\eta = t\langle \eta, \xi \rangle \xi$ where $\xi, \eta \in \mathcal{H}$, $|\xi| = 1$ and $t \neq 0$. Assume $A + B$ is normal. For every continuous function f with compact support in $\mathbf{C}$ the operator $f(A)$ is defined by the spectral theorem. Let $\mathcal{H}_0$ denote the closure of $\{f(A)\xi | f$ continuous of compact support in $\mathbf{C}\}$. Then $\mathcal{H}_0$ is separable and reduces A and $B|\mathcal{H}_0^\perp = 0$. Thus

$$A = A|\mathcal{H}_0 + A|\mathcal{H}_0^\perp, \quad A + B = (A + B)|\mathcal{H}_0 + A|\mathcal{H}_0^\perp.$$

Since B vanishes on $\mathcal{H}_0^\perp$, we shall assume now $\mathcal{H} = \mathcal{H}_0$ i.e. ξ is cyclic for A. Hence A is simple and we may write without loss of generality **[3**, p. 910**]**

$$\mathcal{H} = \mathcal{L}^2(\mathbf{C}, \mu), \quad A = \text{Multiplication by } z.$$

Let λ be an eigenvalue of $A + B$ with normalized eigenvector ζ. Assume λ is also an eigenvector of A with eigenprojection P and normalized eigenvector η. Then $0 = P(A + B - \lambda)\zeta = PB\zeta = t\langle \zeta, \xi \rangle \langle \xi, \eta \rangle \eta$. Since ξ is cyclic for A we have $\langle \xi, \eta \rangle \neq 0$. Hence $\langle \zeta, \xi \rangle = 0$ or $B\zeta = 0$. Thus we get $(A - \lambda)\zeta = 0$.

Since A is simple this is only possible if ζ is a multiple of η, which contradicts $\langle\zeta, \xi\rangle = 0$. Thus we can write $0 = (A + B - \lambda)\zeta = (\lambda - A)(-1 + (\lambda - A)^{-1}B)\zeta$ or $\zeta = (\lambda - A)^{-1}B\zeta = t(\lambda - A)^{-1}\xi\langle\zeta, \xi\rangle$. Taking the scalar product with ξ and observing $\langle\zeta, \xi\rangle \neq 0$ we obtain

$$1 = t\left\langle(\lambda - A)^{-1}\xi, \xi\right\rangle = t\int \frac{|\xi(z)|^2}{(\lambda - z)}\, d\mu(z) \tag{2}$$

COROLLARY 2. *Let A and B as above and let ξ be cyclic for A, i.e. $\mathcal{H} = \mathcal{H}_0$. Then no isolated eigenvalue of A is an eigenvalue of $A + B$, $\sigma(A) \cap \sigma(A + B) = \sigma_{ess}(A)$. Every isolated eigenvalue of A (resp. $A + B$) is simple and the eigenvalues λ of $A + B$ are solutions of* (2).

(2) generalizes (9) of **[4]** and all results of **[4]** and **[5]** can be read off from (2). For example if A is selfadjoint and B positive of finite rank then the eigenvalues of A are moved by B in the positive direction. It is possible to apply the above proof to more general perturbations of finite rank, however the results become less and less tractable.

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