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BUNDLE SHIFTS AND AHLFORS FUNCTIONS

M. B. ABRAHAMSE[1] AND J. J. BASTIAN

ABSTRACT. If S is a bundle shift over R with multiplicity k, if the boundary of R has n components, and if ϕ is the Ahlfors function for a point in R, then $\phi(S)$ is a unilateral shift of multiplicity kn. It follows that a reductive algebra containing the rational algebra of a bundle shift of finite multiplicity is selfadjoint.

Let R be a bounded domain in the complex plane whose boundary consists of a finite number of nonintersecting analytic Jordan curves. A bundle shift over R is a pure subnormal operator with spectrum contained in the closure of R and normal spectrum contained in the boundary of R [**2**]. For a bundle shift S over R and for λ in R, the dimension of the kernel of $(S-\lambda)^*$ is independent of λ and this quantity, either a nonnegative integer or infinity, is said to be the multiplicity of S. The Ahlfors function for a point t in R is the analytic function ϕ which maximizes $\phi'(t)$ within the collection of analytic functions on R bounded by one and having nonnegative derivative at t. The function ϕ extends to be analytic in a neighborhood of the closure of R, it maps the boundary of R onto the unit circle $|z| = 1$, and the number of zeros of ϕ in R (counting multiplicities) is equal to the number of components of the boundary of R [**3**].

THEOREM. *If S is a bundle shift over R with multiplicity k, if the boundary of R has n components, and if ϕ is the Ahlfors function for a point t in R, then $\phi(S)$ is a unilateral shift with multiplicity kn.*

PROOF. It is sufficient to show that (1) the operator $\phi(S)$ is an isometry and (2) the operator $\phi(S)$ is similar to a unilateral shift of multiplicity kn [**2**, Proposition 1.2]. If N is the minimal normal extension of S, then $\phi(N)$ is a unitary extension of $\phi(S)$ which establishes (1). To prove (2), observe that S is similar to the k-fold direct sum $T^{(k)}$ of the operator T on $H^2(R)$ defined by the equation $T(f)(z) = zf(z)$ [**2**, Theorems 1 and 11]. Therefore, the operator $\phi(S)$ is similar to the operator $T_\phi^{(k)}$ where T_ϕ on $H^2(R)$ is defined by the equation $T_\phi(f) = \phi f$. The operator T_ϕ is evidently an isometry, it is pure [**5**, Lemma 3.8], and it has multiplicity n [**1**, Theorem 2.14]. Thus, the operator T_ϕ is a unilateral shift of multiplicity n and this establishes (2).

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An algebra $\mathcal{A}$ of operators is said to be reductive if it is weakly closed, contains the identity, and if every invariant subspace for $\mathcal{A}$ is a reducing subspace for $\mathcal{A}$. The algebra $\mathcal{A}$ is said to be transitive if it is weakly closed, contains the identity, and has no nontrivial invariant subspaces. The reductive algebra problem asks whether every reductive algebra is selfadjoint. The transitive algebra problem asks whether a transitive algebra of operators on a Hilbert space $\mathcal{H}$ must necessarily be the algebra $\mathcal{B}(\mathcal{H})$ of all operators on $\mathcal{H}$. These two problems motivate Corollaries 1 and 2 below. Corollary 3 arises from the problem of determining which operators T have hyperinvariant subspaces, that is, nontrivial subspaces which are invariant for every operator which commutes with T.

For an operator S, let $\mathcal{R}(S)$ denote the algebra of operators $\psi(S)$ where ψ is a rational function with poles off the spectrum of S. Corollaries 1 and 2 follow from the theorem, the corresponding results for unilateral shifts **[4]**, **[6]**, **[7]**, **[8]**, and the fact that the Ahlfors function can be uniformly approximated by rational functions with poles off the closure of R. The proof of Corollary 3 is the same as that of **[8**, Corollary 8.19].

COROLLARY 1. *If a transitive algebra $\mathcal{A}$ on $\mathcal{H}$ contains $\mathcal{R}(S)$ where S is a bundle shift of finite multiplicity, then $\mathcal{A}$ is $\mathcal{B}(\mathcal{H})$.*

COROLLARY 2. *If a reductive algebra contains $\mathcal{R}(S)$ where S is a bundle shift of finite multiplicity, then the algebra is selfadjoint.*

COROLLARY 3. *For $1 \leqslant i, j \leqslant m$, let ϕ_{ij} be a bounded analytic function on R and define T_{ij} on $H^2(R)$ by the equation $T_{ij}(f) = \phi_{ij}f$. If the operator $T = [T_{ij}]$ on $H^2(R)^{(m)}$ is not a scalar multiple of the identity, then T has a hyperinvariant subspace.*

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF VIRGINIA, CHARLOTTESVILLE, VIRGINIA 22903

DEPARTMENT OF MATHEMATICS, WRIGHT STATE UNIVERSITY, DAYTON, OHIO 45435 (Current address of J. J. Bastian)

Current address (M. B. Abrahamse): Department of Mathematics, Loyola University of Chicago, Chicago, Illinois 60626