PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 72, Number 1, October 1978

THE CAUCHY PROBLEM FOR HYPERBOLIC OPERATORS WITH VARIABLE MULTIPLE CHARACTERISTICS

KAZUHIRO YAMAMOTO

ABSTRACT. Let $P(t, x, D_t, D_x)$ be a hyperbolic differential operator with the principal symbol $p_m(t, x, \tau, \xi)$. We assume that P_m is denoted by $\prod_{j=1}^{s}(\tau - \lambda_j)^{m_j}\prod_{j=s+1}^{m-N+s}(\tau - \lambda_j)$ and $(\lambda_i - \lambda_j)(t, x, \xi) \neq 0$ if $(i, j) \neq (k, m - N + k)$ $(k = 1, \dots, s)$, where $N = \sum_{j=1}^{s} m_j$ and $\lambda_j(t, x, \xi) \in C^\infty([0, T] \times R^n \times (R^n \setminus 0))$. Under a generalized condition of E. E. Levi, we shall show that the Cauchy problem $Pu = f$ in $[0, T] \times R^n$, $D_t^j u_{|t=0} = g_j$ $(j = 1, \dots, m - 1)$ is well posed. When $m_j = 1$ $(j = 1, \dots, s)$, our result coincides those of Ohya and Petkov.

1. Statement of the result. We shall consider the following differential operator:

$$P(t, x, D_t, D_x) = \sum_{|\alpha| \leq m} a_\alpha(t, x) D_t^{\alpha_0} D_x^{\alpha'},$$

where $(t, x) \in [0, T] \times R^n$ $(0 < T < \infty)$, $D_t^{\alpha_0} D_x^{\alpha'} = D_t^{\alpha_0} D_{x_1}^{\alpha_1} \cdots D_{x_n}^{\alpha_n}$ and $D_t = -i\partial/\partial t$, $D_{x_j} = -i\partial/\partial x_j$. We assume that $a_{m,0,\dots,0}(t, x) = 1$ and $a_\alpha(t, x)$ belongs to $\mathfrak{B}([0, T] \times R^n)$, which consists of all functions having that these arbitrary derivations are bounded in $[0, T] \times R^n$. Let (τ, ξ) be the covariable of (t, x). Then we define the following symbols:

$$p_k(t, x, \tau, \xi) = \sum_{|\alpha| = k} a_\alpha(t, x) \tau^{\alpha_0} \xi^{\alpha'} \qquad (k = 0, \dots, m).$$

First we shall impose the following assumptions on p_m:

$$p_m(t, x, \tau, \xi) = \prod_{j=1}^{s} (\tau - \lambda_j)^{m_j} \prod_{j=s+1}^{m-N+s} (\tau - \lambda_j), \tag{A.1}$$

where $m_1 \geq m_2 \geq \cdots \geq m_s$, $N = \sum_{j=1}^{s} m_j$ and all functions $\lambda_j(t, x, \xi)$ $(j = 1, \dots, m - N + s)$ are real and positively homogeneous of degree 1 with respect to ξ and belong to $\mathfrak{B}([0, T] \times R^n \times S^{n-1})$ if $\xi \in S^{n-1}$. Here S^{n-1} is the unit sphere of R^n.

(A.2) For any couple $(i, j) \neq (k, m - N + k)$ $(k = 1, \dots, s)$ we suppose the following:

$$|(\lambda_i - \lambda_j)(t, x, \xi)| \geq C, \qquad (t, x, \xi) \in [0, T] \times R^n \times S^{n-1},$$

where C is a positive constant.

Throughout this note, the symbols of pseudo-differential operators are elements of $\mathfrak{B}([0, T] \times R^n \times S^n)$ or $\mathfrak{B}([0, T] \times R^n \times S^{n-1})$ if $(\tau, \xi) \in S^n$ or

Received by the editors May 13, 1977 and, in revised form, August 8, 1977.
AMS (MOS) subject classifications (1970). Primary 35L30; Secondary 35L45.

$\xi \in S^{n-1}$. Moreover we use the following notation. For symbols $a(t, x, \xi)$, $b(t, x, \xi)$ and $A(t, x, \tau, \xi)$, $A_{|\tau=a} \equiv 0 \bmod b$ means that there exists a symbol $c(t, x, \xi)$ such that

$$A(t, x, a(t, x, \xi), \xi) = c(t, x, \xi)b(t, x, \xi).$$

For the lower order terms of P we assume the following:

(A.3) For any k $(k = 1, \dots, s)$ we can denote $P(t, x, D_t, D_x)$ by the following form:

$$P = \sum_{l=0}^{m_k} Q_{k,l}(t, x, D_t, D_x)(\Lambda_k(t, x, D_t, D_x))^l. \tag{1.1}$$

Here Λ_k is the pseudo-differential operator defined by the symbol $\tau - \lambda_k(t, x, \xi)$ and $Q_{k,l}$ $(l = 0, \dots, m_k)$ is a pseudo-differential operator of order $m - m_k$, whose principal symbol $q_{k,l}(t, x, \tau, \xi)$ has the following property:

$$q_{k,l|\tau=\lambda_k} \equiv 0 \quad \bmod \lambda_k - \lambda_{m-N+k}. \tag{1.2}$$

Clearly if $\lambda_k = \lambda_{m-N+k}$, then the above condition (A.3) is that of E. E. Levi. In the final part of this note we denote (A.3) by the condition with respect to p_k, when $m_k = 1$ or 2.

For a nonnegative integer k and $s \in R$ the function space $C^k([0, T]; H_s(R^n))$ consists of functions such that $D_t^j u(t)$ $(j = 0, \dots, k)$ exists as an element of $H_{s-j}(R^n)$ and is continuous on the topology of $H_{s-j}(R^n)$. We use the following norm:

$$|||u(t)|||_{s,k}^2 = \sum_{j=0}^{k} \|D_t^j u(t)\|_{s-j}^2,$$

where $\|\cdot\|_{s-j}$ is the usual norm of $H_{s-j}(R^n)$.

Now we can state our theorem.

THEOREM. *Let $P(t, x, D_t, D_x)$ be a differential operator of order m. If P satisfies the assumptions* (A.1), (A.2) *and* (A.3), *then the Cauchy problem $Pu = f$ in $[0, T] \times R^n$, $D_t^j u_{|t=0} = g_j$ $(j = 0, \dots, m-1)$ is well posed, i.e., for $f \in C^{k-m+m_1+1}([0, T]; H_{s-m+m_1+1}(R^n))$ and $g_j \in H_{s-j+m_1}(R^n)$ there exists a unique solution $u(t, x) \in C^k([0, T]; H_s(R^n))$ such that*

$$|||u(t)|||_{s,k} \leqslant C\left\{\sum_{j=0}^{m-1} \|g_j\|_{s-j+m_1} + |||f(0)|||_{s-m+m_1,k-m+m_1} + \int_0^t |||f(r)|||_{s-m+m_1+1,k-m+m_1+1} dr\right\},$$

where $k \geqslant m - m_1 - 1$, $|||f(0)|||_{s-m+m_1,-1} = 0$ and $t \in [0, T]$.

This Theorem is the same as those of [3] and [4] when $m_j = 1$ $(j = 1, \dots, s)$. Under a different situation, in [**1**] they consider the Cauchy problem of a triple case.

2. Reform of the condition (A.3). In this section, we state an equivalent condition of (A.3).

Taking care of multiplicities of the roots λ_j ($j = 1, \ldots, s$), we can denote $p_m(t, x, \tau, \xi)$ by

$$(\tau - \lambda_{m-N+s}) \cdots (\tau - \lambda_{s+1}) \Phi_\mu^{n_\mu} \cdots \Phi_1^{n_1},$$

where $\Phi_\nu(t, x, \tau, \xi)$ ($\nu = 1, \ldots, \mu$) is a polynomial of degree s with respect to τ and equal to $\prod_{j=1}^{s_\nu}(\tau - \lambda_j)$. Here $m_{s_{\nu-1}+1} = \cdots = m_{s_\nu}$ ($\nu = 1, \ldots, \mu$). Remark that $s_1 < s_2 < \cdots < s_\mu = s$, $m_1 = \sum_{\nu=1}^{\mu} n_\nu$ and denote N_ν by $s_\nu n_\nu$. We introduce a product pseudo-differential operator $\Phi_\nu(t, x, D_t, D_x) = (\Lambda_{s_\nu} \cdots \Lambda_1)(t, x, D_t, D_x)$. Then we denote $\Delta_j(t, x, D_t, D_x)$ of order j ($j = 0, \ldots, m$) by $\Delta_0 = 1$, $\Delta_1 = \Lambda_1, \ldots, \Lambda_j, \ldots, \Delta_N = \Phi_\mu^{n_\mu} \cdots \Phi_1^{n_1}, \ldots, \Delta_{N+k} = \Lambda_{s+k} \cdots \Lambda_{s+1}\Delta_N, \ldots, \Delta_m = \Lambda_{m-N+s} \cdots \Lambda_{s+1}\Delta_N$, where $\Delta_j = \Lambda_\delta \cdots \Lambda_1 \Phi_\nu^\sigma \Phi_{\nu-1}^{n_{\nu-1}} \cdots \Phi_1^{n_1}$ if $j = (N_1 + \cdots + N_{\nu-1}) + \sigma s_\nu + \delta$ ($\sigma = 0, \ldots, n_\nu - 1$, $\delta = 0, \ldots, s_{\nu-1}$). Then we have the following:

PROPOSITION 2.1. *Let $P(t, x, D_t, D_x)$ be a differential operator which satisfies the conditions* (A.1) and (A.2). *Then the condition* (A.3) *is equivalent to the following statement. We can denote P by*

$$P(t, x, D_t, D_x) = \sum_{i=0}^{m_1} Q_i(t, x, D_t, D_x)\Delta_{I(i)}, \tag{2.1}$$

where if $i = n_\mu + \cdots + n_{\nu+1} + \sigma$ ($1 \leqslant \sigma \leqslant n_\nu$), then $I(i) = N_1 + \cdots + N_\nu - s_\nu\sigma$ and Q_i ($i = 0, \ldots, m_1$) is a pseudo-differential operator of order $M_i = m - i - I(i)$ and differential operator of t. Moreover the principal symbol $q_i(t, x, \tau, \xi)$ of Q_i satisfies the following condition:

$$q_{i|\tau=\lambda_k} \equiv 0 \mod \lambda_k - \lambda_{m-N+k} \quad \text{if } k \leqslant s_\nu. \tag{2.2}$$

Since any pseudo-differential operator of order l ($\leqslant m$) which is a differential operator of t is represented by $\Delta_0, \ldots, \Delta_m$, we have the following:

PROPOSITION 2.2. *Let P be a differential operator which satisfies the conditions* (A.1), (A.2), (2.1) *and* (2.2). *Then P is denoted by*

$$P = \sum_{i=0}^{m_1} \sum_{j=0}^{M_i} R_{i,j}(t, x, D_x)\Delta_{I(i)+j}, \tag{2.3}$$

where $R_{i,j}$ is a pseudo-differential operator of order $M_i - j$ whose principal symbol is $r_{i,j}(t, x, \xi)$, and $r_{i,j}$ have the following property:

$$\sum_{j=0}^{k-1} r_{i,j}\Lambda_j \cdots \Lambda_{1|\tau=\lambda_k} \equiv 0 \mod \lambda_k - \lambda_{m-N+k}, \qquad k \leqslant s_\nu, \tag{2.4}$$

if $i = n_\mu + \cdots + n_{\nu+1} + \sigma$ ($1 \leqslant \sigma \leqslant n_\nu$).

3. Reduction to a first order system and the proof of the Theorem. Since the proof of the Theorem is inferred on the analogy of a simple case, we assume that $s = 2$, $m_1 = 2$ and $m_2 = 1$. Thus by (2.3) our considered operator

$P(t, x, D_t, D_x)$ in (1.1) is denoted by

$$P(t, x, D_t, D_x) = \Delta_m + \sum_{i=1}^{2} \sum_{j=0}^{m-i} R_j^{m-i}\Delta_j,$$

where R_j^{m-i} is of order $m - i - j$. The condition (2.4) says that

$$R_0^{m-1}(t, x, D_x) = 0, \tag{3.1}$$

$$r_1^{m-1}(t, x, \xi) \equiv r_0^{m-2}(t, x, \xi) \equiv 0 \mod \lambda_1 - \lambda_{m-2}, \tag{3.2}$$

$$\left(r_1^{m-1} + r_2^{m-1}(\lambda_2 - \lambda_1)\right)(t, x, \xi) \equiv 0 \mod \lambda_2 - \lambda_{m-1}. \tag{3.3}$$

We denote a pseudo-differential operator with the symbol $|\xi|$ by the term Λ and define a column vector

$$U = {}^t\left(\Lambda^{m-3}\Delta_0 u, \Lambda^{m-3}\Delta_1 u, \Lambda^{m-4}\Delta_2 u, \Lambda^{m-4}\Delta_3 u, \ldots, \Lambda\Delta_{m-2}u, \Delta_{m-1}u\right) \tag{3.4}$$

and $F = {}^t(0, \ldots, 0, f)$. Then by (3.1) the equation $Pu = f$ becomes the following first order system:

$$MU = (D_t - A(t, x, D_x))U + B(t, x, D_x)U = F,$$

where B is of order 0 and A is a first order pseudo-differential operator with the symbol

$$a(t, x, \xi) = \begin{pmatrix} \lambda_1, 0, & & & & & \\ & \lambda_1, |\xi|, & & & \bigcirc & \\ & & \lambda_2, 0, & & & \\ & & & \lambda_3, |\xi|, & & \\ & \bigcirc & & & & \\ & & & & ,\lambda_{m-2}, |\xi| & \\ a_{m1}, a_{m2}, a_{m3}, 0, & & , 0 & & & ,\lambda_{m-1} \end{pmatrix},$$

where $a_{m1} = -r_0^{m-2}(t, x, \xi/|\xi|)|\xi|$, $a_{m2} = -r_1^{m-1}(t, x, \xi/|\xi|)|\xi|$ and $a_{m3} = -r_2^{m-1}(t, x, \xi/|\xi|)|\xi|$. Then a has the following property.

PROPOSITION 3.1. *There exists a nonsingular matrix $N(t, x, \xi)$ whose components are homogeneous of degree* 0 *and belong to* $\mathfrak{B}([0, T] \times R^n \times (R^n \setminus 0))$ *such that*

$$(aN)(t, x, \xi) = (ND)(t, x, \xi),$$

where $D = (d_{ij})$ is a $m \times m$ diagonal matrix with $d_{11} = d_{22} = \lambda_1$ and $d_{ii} = \lambda_{i-1}$ $(i = 2, \ldots, m)$.

PROOF. Since clearly an eigenvalue of $a(t, x, \xi/|\xi|)$ is $\tilde{\lambda}_j = \lambda_j(t, x, \xi/|\xi|)$ $(j = 1, \ldots, m-1)$, we shall seek an eigenvector of $\tilde{\lambda}_j$. The equation

$$(\mu I - a(t, x, \xi/|\xi|))n = 0,$$

where $n = {}^t(n_1, \ldots, n_m)$, is equivalent to the following:

$$\begin{gathered}(\mu - \tilde{\lambda}_1)n_1 = 0, \quad (\mu - \tilde{\lambda}_1)n_2 - n_3 = 0, \quad (\mu - \tilde{\lambda}_2)n_3 = 0, \\ (\mu - \tilde{\lambda}_3)n_4 = n_5, \ldots, (\mu - \tilde{\lambda}_{m-2})n_{m-1} = n_m, \\ (\mu - \tilde{\lambda}_{m-1})n_m + \tilde{r}_0^{m-2}n_1 + \tilde{r}_1^{m-1}n_2 + \tilde{r}_2^{m-1}n_3 = 0, \end{gathered} \tag{3.5}$$

where $\tilde{r}_j^{m-i}(t, x, \xi) = r_j^{m-i}(t, x, \xi/|\xi|)$. Since $\tilde{r}_1^{m-1}n_2 + \tilde{r}_2^{m-1}n_3 = (\tilde{r}_1^{m-1} + \tilde{r}_2^{m-1}(\mu - \lambda_1))n_2$, for example in fact we can take the following matrix $N(t, x, \xi)$ such that each column of N is an eigenvector of an eigenvalue $\tilde{\lambda}_1$, $\tilde{\lambda}_1, \tilde{\lambda}_2, \ldots, \tilde{\lambda}_{m-1}$ respectively in turn;

$$N(t, x, \xi) = \begin{pmatrix} 1 & ,0 & ,0 & ,0, & \cdots & ,0 \\ 0 & ,1 & ,(\lambda_2 - \lambda_1)^{-1} & ,0, & \cdots & ,0 \\ 0 & ,0 & ,1 & ,0, & \cdots & ,0 \\ & & & ,1 & & \\ & & * & & \ddots & * \\ n_{m-11}, & n_{m-12}, & n_{m-13} & , & 0 & \ddots \\ n_{m1} & ,n_{m2} & ,n_{m3} & , & & ,1 \end{pmatrix}$$

where

$$\begin{aligned} n_{m1} &= \tilde{r}_0^{m-2}/(\tilde{\lambda}_1 - \tilde{\lambda}_{m-1}), \quad n_{m-11} = \tilde{r}_0^{m-2}/(\tilde{\lambda}_1 - \tilde{\lambda}_{m-1})(\tilde{\lambda}_1 - \tilde{\lambda}_{m-2}), \\ n_{m2} &= \tilde{r}_1^{m-1}/(\tilde{\lambda}_1 - \tilde{\lambda}_{m-1}), \quad n_{m-12} = \tilde{r}_1^{m-1}/(\tilde{\lambda}_1 - \tilde{\lambda}_{m-1})(\tilde{\lambda}_1 - \tilde{\lambda}_{m-2}), \\ n_{m3} &= (\tilde{r}_1^{m-1} + \tilde{r}_2^{m-1}(\tilde{\lambda}_2 - \tilde{\lambda}_1))/(\tilde{\lambda}_2 - \tilde{\lambda}_{m-1}), \quad n_{m-13} = n_{m3}/(\tilde{\lambda}_2 - \tilde{\lambda}_{m-2}). \end{aligned}$$

These are well defined by (A.2), (3.2) and (3.3) and * are easily inductively determined from (A.2) and (3.5). Clearly the matrix $N(t, x, \xi)$ is positively homogeneous of degree 0 and a nonsingular matrix in $[0, T] \times R^n \times (R^n \setminus 0)$. This completes the proof of Proposition 3.1.

By Proposition 3.1 the following fact is well known (see [2]), which guarantees the validity of our theorem.

PROPOSITION 3.2. *For any nonnegative integer k and $s \in R$ if $F(t, x) \in C^k([0, T]; H_s(R^n))$ and $U_0(x) \in H_s(R^n)$, then there exists a unique solution $U(t, x) \in C^k([0, T]; H_s(R^n))$ of the Cauchy problem for M with initial data on $t = 0$ such that*

$$|||U(t)|||_{s,k} \leq C\left\{|||U(0)|||_{s,k} + \int_0^t |||MU(r)|||_{s,k}dr\right\},$$

where C does not depend on $U(t, x)$ and $t \in [0, T]$.

REMARK. For a general case the vector U of (3.4) becomes the following:

$$U = {}^t\left(\Lambda^{m-m_1-1}\Delta_0 u, \ldots, \Lambda^{a_j}\Delta_j, \ldots, \Lambda^{m-N-1}\Delta_N, \ldots, \Lambda\Delta_{m-2}, \Delta_{m-1}\right),$$

where $a_j = m - (n_\nu + \cdots + n_\mu) + \sigma - j - 1$ if $j = N_1 + \cdots + N_{\nu-1} + s_\nu\sigma + \delta$ ($\sigma = 0, \ldots, n_\nu - 1, \delta = 0, \ldots, s_\nu - 1$).

4. Some sufficient condition of (A.3) when $m_k = 1$ or 2. In this section we shall denote (A.3) by the conditions with respect to p_k when $m_k = 1$ or 2. We denote the subprincipal symbol of P by

$$p^s_{m-1}(x, \xi) = p_{m-1} + (i/2)\sum_{j=0}^{n} \frac{\partial^2 p_m}{\partial x_j \partial \xi_j}$$

and the Poisson bracket of $f(x, \xi)$ and $g(x, \xi)$ by

$$\{f, g\}(x, \xi) = \sum_{j=0}^{n}\left(\frac{\partial f}{\partial \xi_j}\frac{\partial g}{\partial x_j} - \frac{\partial f}{\partial x_j}\frac{\partial g}{\partial \xi_j}\right)(x, \xi),$$

where for simplicity we identify (t, τ) with (x_0, ξ_0) in this section. We state the following sufficient condition of (A.3), which is a generalized condition of that in [4].

PROPOSITION 4.1. *We have the following*:

(i) *When $m_k = 1$, the condition* (A.3) *is equivalent to the following*:

$$p^s_{m-1} + (i/2)r_0\{\Lambda_{m-N+k}, \Lambda_k\}_{|\tau=\lambda_k} \equiv 0 \mod \lambda_k - \lambda_{m-N+k}, \tag{4.1}$$

where $r_0(t, x, \tau, \xi) = p_m/(\tau - \lambda_k)(\tau - \lambda_{m-N+k})$.

(ii) *When $m_k = 2$, if the following three conditions hold, then* (A.3) *is valid*;

$$p^s_{m-1}(t, x, \lambda_k(t, x, \xi), \xi) = 0, \tag{4.2}$$

$$\partial p^s_{m-1}/\partial\tau_{|\tau=\lambda_k} \equiv \{\Lambda_{m-N+k}, \Lambda_k\}_{|\tau=\lambda_k} \equiv 0 \mod \lambda_k - \lambda_{m-N+k}, \tag{4.3}$$

$$\begin{aligned} p_{m-2} &- \sum_{j=0}^{n} p^{(j)}_{m-1,j}/2 + \sum_{l,j=0}^{n} p^{(lj)}_{m,lj}/8 \\ &- \sum_{j=0}^{n} r_0\left(\{\Lambda^{(j)}_{m-N+k}, \Lambda_k\}\Lambda_{k,j} - \{\Lambda_{m-N+k,j}, \Lambda_k\}\Lambda^{(j)}_k\right)/4 \equiv 0 \\ &\mod \lambda_k - \lambda_{m-N+k}, \end{aligned} \tag{4.4}$$

where $f^{(\alpha)}_\beta = (iD^\alpha_\xi)D^\beta_x f(x, \xi)$ and $r_0(t, x, \tau, \xi) = p_m/(\tau - \lambda_k)^2(\tau - \lambda_{m-N+k})$.

PROOF. (i) Since $R_0\Lambda_k$, where R_0 is a pseudo-differential operator with the symbol $p_m/(\tau - \lambda_k)$, satisfies the condition (A.3), we may show that the principal symbol $r(x, \xi)$ of $P - R_0\Lambda_k$ satisfies the condition: $r_{|\tau=\lambda_k} \equiv 0 \mod \lambda_k - \lambda_{m-N+k}$. By an easy computation we see that

$$(r - p^s_{m-1}) - \frac{i}{2} r_0\{\Lambda_{m-N+k}, \Lambda_k\}_{|\tau=\lambda_k} \equiv 0 \mod \lambda_k - \lambda_{m-N+k}.$$

This implies (4.1).

(ii) In this case if we denote the principal symbol of $P - R_0\Lambda_k^2$ by $r(t, x, \tau, \xi)$, where R_0 is a pseudo-differential operator defined by the symbol $p_m/(\tau - \lambda_k)^2$, then r is written by the following form:

$$r = p_{m-1}^s + \left(ir_0\{\Lambda_{m-N+k}, \Lambda_k\} + A\Lambda_{m-N+k}\right)\Lambda_k, \tag{4.5}$$

where A is some positively homogeneous function of degree $m - 3$. Thus by (4.2) and (4.5) we can denote P by $R_0\Lambda_k^2 + R_1\Lambda_k + R_2$, where R_j is a pseudo-differential operator of order $m - 2$ and the principal symbol is written by $r_j(t, x, \tau, \xi)$. By (4.5) if we assume (4.3), then we have $r_{1|\tau=\lambda_k} \equiv 0$ mod $\lambda_k - \lambda_{m-N+k}$.

For simplicity we denote the homogeneous symbol of order i of a pseudo-differential operator $Q(t, x, D_t, D_x)$ by $\sigma_i(Q)$. Then we have

$$r_{2|\tau=\lambda_k} = p_{m-2} - \sigma_{m-2}\left(R_0\Lambda_k^2\right) - \sum_{j=0}^{n} r_1^{(j)}\Lambda_{k,j|\tau=\lambda_k}. \tag{4.6}$$

Since (4.3) holds and $r_1\Lambda_k = p_{m-1} - \sigma_{m-1}(R_0\Lambda_k^2)$, we have

$$\sum_{j=0}^{n}\left(r_1^{(j)}\Lambda_{k,j} - \left(p_{m-1} - \sigma_{m-1}\left(R_0\Lambda_k^2\right)\right)_j^{(j)}/2\right)_{|\tau=\lambda_k} \equiv 0 \mod \lambda_k - \lambda_{m-N+k}.$$

Therefore we have

$$r_{2|\tau=\lambda_k} \equiv p_{m-2} - \sum_{j=0}^{n} p_{m-1,j}^{(j)}/2 + \sum_{j=0}^{n} \sigma_{m-1}\left(R_0\Lambda_k^2\right)_j^{(j)}/2$$
$$- \sigma_{m-2}\left(R_0\Lambda_k^2\right)_{|\tau=\lambda_k} \mod \lambda_k - \lambda_{m-N+k}. \tag{4.7}$$

Here

$$\sigma_{m-1}\left(R_0\Lambda_k^2\right) = \sum_{j=0}^{n}\left(r_0\Lambda_k^{(j)}\Lambda_{k,j} + 2r_0^{(j)}\Lambda_{k,j}\Lambda_k\right), \tag{4.8}$$

$$\sigma_{m-2}\left(R_0\Lambda_k^2\right)_{|\tau=\lambda_k} \equiv \sum_{l,j=0}^{n}\left(r_0^{(l)}\left(\Lambda_{k,l}^{(j)}\Lambda_{k,j} + \Lambda_k^{(j)}\Lambda_{k,lj}\right) + r_0^{(lj)}\Lambda_{k,l}\Lambda_{k,j}\right)_{|\tau=\lambda_k}$$
$$\mod \lambda_k - \lambda_{m-N+k}. \tag{4.9}$$

By taking care of (4.8), (4.9) and (4.3), add $\sum_{l,j=0}^{n} p_{m,lj}^{(lj)}/8_{|\tau=\lambda_k}$ to the right hand side of (4.7). Then we have that $r_{2|\tau=\lambda_k}$ is equal to the left hand side of (4.4) mod $\lambda_k - \lambda_{m-N+k}$. This completes the proof of Proposition 4.1.

References

1. K. Kitagawa and T. Sadamatsu, *Une condition suffisante pour que le problème de Cauchy hyperbolique soit bien posé*, J. Math. Kyoto Univ. **17** (1977), 467–499.

2. S. Mizohata, *The theory of partial differential equations*, Cambridge Univ. Press, London and New York, 1973.

3. Y. Ohya, *Le problème de Cauchy à caractéristiques multiples*, Ann. Scuola Norm. Sup. Pisa (to appear).

4. V. M. Petkov, *The Cauchy problem for a class of non-strictly hyperbolic equations with double characteristics*, Serdica, Bulg. Math. Publ. 1 (1975), 372–380.

Department of Mathematics, Faculty of Science, Hokkaido University, Sapporo 060, Japan