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CONVERGENCE OF CERTAIN COSINE SUMS IN A METRIC SPACE $-$ L

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ABSTRACT. In this paper a generalization of a theorem of Garrett and Stanojevic [6] has been obtained.

1. A sequence $\{a_n\}$ is said to be convex if $\Delta^2 a_n \geqslant 0$, where $\Delta^2 a_n = \Delta a_n - \Delta a_{n+1}$ and $\Delta a_n = a_n - a_{n+1}$. It is said to be quasi-convex if

$$\sum_{n=1}^{\infty} (n+1)|\Delta^2 a_n| < \infty.$$

It is well known that every bounded convex sequence is quasi-convex but the converse is not always true. The concept of null quasi-convex sequence has been further generalized by Sidon [1] in the following manner.

A sequence $\{a_n\}$ is said to satisfy condition S if

(i) $a_n = 0(1)$, $n \to \infty$,

(ii) there exist numbers A_n such that $A_n \underset{0}{\downarrow}$ and $\sum A_n < \infty$,

(iii) $|\Delta a_n| \leqslant A_n$ for all n.

By taking $A_n = \sum_{m=n}^{\infty} |\Delta^2 a_n|$, we observe that every null quasi-convex sequence $\{a_n\}$ belongs to class S. As regards the converse, it is clear from the example $a_n = (-1)^n/n$, that a sequence $\{a_n\} \in S$ need not necessarily be quasi-convex.

A sequence of nonnegative numbers is said to be quasi-monotone if $a_{n+1} \leqslant a_n(1 + \alpha/n)$ for some constant $\alpha > 0$ and all $n > n_0(\alpha)$.

An equivalent definition is that $n^{-\beta} a_n \underset{0}{\downarrow}$ for same $\beta > 0$.

A sequence $\{a_n\}$ is said to satisfy condition S' if

(i) $a_n = 0(1)$, $n \to \infty$,

(ii) there exist numbers A_n such that $\{A_n\}$ is a quasi-monotone sequence and $\sum_{n=1}^{\infty} A_n < \infty$,

(iii) $|\Delta a_n| \leqslant A_n$ for all n.

It is clear that the condition S' is weaker than the condition S.

2. Let

$$\frac{1}{2} a_0 + \sum_{n=1}^{\infty} a_n \cos nx \text{ be a trigonometric series.} \tag{2.1}$$

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Let

$$f(x) = \frac{1}{2} a_0 + \sum_{n=1}^{\infty} a_n \cos nx,$$

and

$$f_n(x) = \frac{1}{2} \sum_{k=0}^{n} \Delta a_k + \sum_{k=1}^{n} \left(\sum_{j=k}^{n} \Delta a_j \right) \cos kx.$$

Concerning the convergence of $f_n(x)$ in L-metric, Garrett and Stanojevic **[6]** proved the following theorem:

THEOREM A. *If $\{a_n\}$ is a null quasi-convex sequence, then $f_n(x)$ converges to $f(x)$ in L-metric.*

Babu Ram [9] and Maher M. H. Marzug [7] proved that the condition S also implies the conclusion of the Garrett-Stanojevic theorem.

The aim of this paper is to generalize this theorem and those of Babu Ram and Marzug by using condition S' in place of null quasi-convex sequence or condition S. We prove the following theorem.

THEOREM. *Let $a_n \in S'$, then $f_n(x)$ converges to $f(x)$ in L - metric.*

3. For the proof of this theorem we require the following lemmas:

LEMMA ([1], [3]). *If the numbers α_i, $i = 0, 1, \ldots, k$, satisfy the condition $|\alpha_i| \leqslant 1$, then the following estimate is valid*:

$$\int_0^{\pi} \left| \sum_{i=0}^{k} \frac{\alpha_i \sin(i + 1/2)x}{2 \sin x/2} \right| dx \leqslant C(k+1),$$

where C is an absolute constant.

LEMMA 2 [2]. *Let $\{a_n\}$ be a quasi-monotone sequence of constants. If $\sum_{n=1}^{\infty} a_n$ converges, then $na_n = o(1)$.*

LEMMA 3. *Let $\{a_n\}$ be a quasi-monotone sequence of constants. If $\sum_{n=1}^{\infty} a_n$ converges, then $\sum_{n=1}^{\infty}(n+1)|\Delta a_n|$ converges.*

The proof is easy.

REMARK. This lemma also follows as a particular case of a more general theorem by Robertson **[4]**.

THEOREM. *If the sequence $\{a_n\}$ is (ϕ, δ) monotone and $\sum a_n \Delta \phi_n$ converges, then the series $\sum \phi_{n+1} \Delta a_n$ is absolutely convergent.*

PROOF OF THE THEOREM. By Abel's transformation

$$f(x) = \lim_{n\to\infty}\left[\frac{1}{2}a_0 + \sum_{k=1}^{n} a_k \cos kx\right]$$

$$= \lim_{n\to\infty}\left[\frac{1}{2}a_0 + \sum_{k=1}^{n-1} D_k(x)\Delta a_k + a_n D_n(x) - \frac{1}{2}a_0\right]$$

$$= \lim_{n\to\infty}\left[\sum_{k=0}^{n-1} D_k(x)\Delta a_k + a_n D_n(x)\right]$$

$$= \sum_{k=0}^{\infty} D_k(x)\Delta a_k, \quad \text{since } \lim_{n\to\infty} a_n D_n(x) = 0 \quad \text{if } x \neq 0,$$

where $D_n(x) = 1/2 + \cos x + \cos 2x + \cdots + \cos nx$.

Again, by Abel's transformation,

$$f_n(x) = \frac{1}{2}\sum_{k=0}^{n}\Delta a_k + \sum_{k=1}^{n}\left(\sum_{j=k}^{n}\Delta a_j\right)\cos kx$$

$$= \sum_{k=0}^{n} D_k(x)\Delta a_k.$$

Now

$$\int_0^\pi |f(x) - f_n(x)|dx = \int_0^\pi \left|\sum_{k=n+1}^{\infty} D_k(x)\Delta a_k\right| dx$$

$$= \lim_{N\to\infty}\int_0^\pi\left|\sum_{k=n+1}^{N} D_k(x)\Delta a_k\right|dx = \lim_{N\to\infty}\int_0^\pi\left|\sum_{k=n+1}^{N} A_k D_k(x)\frac{\Delta a_k}{A_k}\right|dx$$

$$= \lim_{N\to\infty}\int_0^\pi\left|\sum_{k=n+1}^{N-1}\Delta A_k\sum_{\mu=1}^{k} D_\mu(x)\frac{\Delta a_\mu}{A_\mu}\right.$$

$$\left. + A_N\sum_{k=1}^{N} D_k(x)\frac{\Delta a_k}{A_k} - A_{n+1}\sum_{k=1}^{n} D_k(x)\frac{\Delta a_k}{A_k}\right|dx$$

$$\leqslant \lim_{N\to\infty}\left[\sum_{k=n+1}^{N-1}|\Delta A_k|\int_0^\pi\left|\sum_{\mu=1}^{k}\frac{\Delta a_\mu}{A_\mu}D_\mu(x)\right|dx\right.$$

$$\left. + A_N\int_0^\pi\left|\sum_{k=1}^{N}\frac{\Delta a_k}{A_k}D_k(x)\right|dx + A_{n+1}\int_0^\pi\left|\sum_{k=1}^{n}\frac{\Delta a_k}{A_k}D_k(x)\right|dx\right]$$

$$\leqslant \lim_{N\to\infty} C\left[\sum_{k=n+1}^{N-1}(k+1)|\Delta A_k| + A_N(N+1) + A_{n+1}(n+1)\right],$$

by Lemma 1, since $|\Delta a_k/A_k| \leqslant 1$ for each k;

$$\leqslant C\sum_{k=n+1}^{\infty}(k+1)|\Delta A_k| + (n+1)A_{n+1},$$

since by Lemma 2

$$(N+1)A_N = 0(1), \quad \text{as } N \to \infty.$$

But by Lemma 3, $\sum_{k=1}^{\infty}(k+1)|\Delta A_k|$ converges, so by Lemmas 2 and 3 we have

$$\lim_{n\to\infty} \int_0^{\pi} |f(x) - f_n(x)|dx = 0(1).$$

This proves the theorem.

Note. This theorem gives a direct proof that the condition S' implies the necessary and sufficient condition of Garrett-Stanojevic [**8**, Theorem 1].

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