CONVERGENCE OF CERTAIN COSINE SUMS IN A METRIC SPACE — L

NIRANJAN SINGH AND K. M. SHARMA

Abstract. In this paper a generalization of a theorem of Garrett and Stanojevic [6] has been obtained.

1. A sequence \( \{a_n\} \) is said to be convex if \( \Delta^2 a_n > 0 \), where \( \Delta^2 a_n = \Delta a_n - \Delta a_{n+1} \) and \( \Delta a_n = a_n - a_{n+1} \). It is said to be quasi-convex if

\[
\sum_{n=1}^{\infty} (n + 1)|\Delta^2 a_n| < \infty.
\]

It is well known that every bounded convex sequence is quasi-convex but the converse is not always true. The concept of null quasi-convex sequence has been further generalized by Sidon [1] in the following manner.

A sequence \( \{a_n\} \) is said to satisfy condition \( S \) if

(i) \( a^n = O(1) \), \( n \to \infty \),

(ii) there exist numbers \( A_n \) such that \( A_n \downarrow \) and \( \Sigma A_n < \infty \),

(iii) \( |\Delta a^n| < A_n \) for all \( n \).

By taking \( A_n = \Sigma_{m=n}^{\infty} |\Delta^2 a_m| \), we observe that every null quasi-convex sequence \( \{a^n\} \) belongs to class \( S \). As regards the converse, it is clear from the example \( a_n = (-1)^n \), that a sequence \( \{a^n\} \in S \) need not necessarily be quasi-convex.

A sequence of nonnegative numbers is said to be quasi-monotone if \( a_{n+1} < a_n(1 + \alpha/n) \) for some constant \( \alpha > 0 \) and all \( n > n_0(\alpha) \).

An equivalent definition is that \( n^{-\beta} A_n \downarrow \) for some \( \beta > 0 \).

A sequence \( \{a_n\} \) is said to satisfy condition \( S' \) if

(i) \( a_n = O(1) \), \( n \to \infty \),

(ii) there exist numbers \( A_n \) such that \( \{A_n\} \) is a quasi-monotone sequence and \( \Sigma_{n=1}^{\infty} A_n = 0 \),

(iii) \( |\Delta a_n| < A_n \) for all \( n \).

It is clear that the condition \( S' \) is weaker than the condition \( S \).

2. Let

\[
\frac{1}{2} a_0 + \sum_{n=1}^{\infty} a_n \cos nx
\]

be a trigonometric series. (2.1)
Let
\[ f(x) = \frac{1}{2} a_0 + \sum_{n=1}^{\infty} a_n \cos nx, \]
and
\[ f_n(x) = \frac{1}{2} \sum_{k=0}^{n} \Delta a_k + \sum_{k=1}^{n} \left( \sum_{j=k}^{n} \Delta a_j \right) \cos kx. \]

Concerning the convergence of \( f_n(x) \) in \( L \)-metric, Garrett and Stanojevic [6] proved the following theorem:

**Theorem A.** If \( \{a_n\} \) is a null quasi-convex sequence, then \( f_n(x) \) converges to \( f(x) \) in \( L \)-metric.

Babu Ram [9] and Maher M. H. Marzug [7] proved that the condition \( S \) also implies the conclusion of the Garrett-Stanojevic theorem.

The aim of this paper is to generalize this theorem and those of Babu Ram and Marzug by using condition \( S' \) in place of null quasi-convex sequence or condition \( S \). We prove the following theorem.

**Theorem.** Let \( a_n \in S' \), then \( f_n(x) \) converges to \( f(x) \) in \( L \)-metric.

3. For the proof of this theorem we require the following lemmas:

**Lemma (1), (3).** If the numbers \( a_i, \ i = 0, 1, \ldots, k, \) satisfy the condition \( |a_i| < 1 \), then the following estimate is valid:
\[ \int_0^\pi \left| \sum_{i=0}^{k} a_i \sin(i + 1/2)x \right| \frac{2 \sin x/2}{2 \sin x/2} \, dx \leq C (k + 1), \]
where \( C \) is an absolute constant.

**Lemma 2 [2].** Let \( \{a_n\} \) be a quasi-monotone sequence of constants. If \( \sum_{n=1}^{\infty} a_n \) converges, then \( na_n = o(1) \).

**Lemma 3.** Let \( \{a_n\} \) be a quasi-monotone sequence of constants. If \( \sum_{n=1}^{\infty} a_n \) converges, then \( \sum_{n=1}^{\infty} (n + 1)|\Delta a_n| \) converges.

The proof is easy.

**Remark.** This lemma also follows as a particular case of a more general theorem by Robertson [4].

**Theorem.** If the sequence \( \{a_n\} \) is \((\phi, \delta)\) monotone and \( \Sigma a_n \Delta \phi_n \) converges, then the series \( \Sigma \phi_{n+1} \Delta a_n \) is absolutely convergent.

**Proof of the theorem.** By Abel's transformation
\[
f(x) = \lim_{n \to \infty} \left[ \frac{1}{2} a_0 + \sum_{k=1}^{n} a_k \cos kx \right]
\]
\[
= \lim_{n \to \infty} \left[ \frac{1}{2} a_0 + \sum_{k=1}^{n-1} D_k(x) \Delta a_k + a_n D_n(x) - \frac{1}{2} a_0 \right]
\]
\[
= \lim_{n \to \infty} \sum_{k=0}^{n-1} D_k(x) \Delta a_k + a_n D_n(x)
\]
\[
= \sum_{k=0}^{\infty} D_k(x) \Delta a_k, \quad \text{since } \lim_{n \to \infty} a_n D_n(x) = 0 \text{ if } x \neq 0,
\]
where \( D_k(x) = 1/2 + \cos x + \cos 2x + \cdots + \cos nx. \)

Again, by Abel's transformation,
\[
f_n(x) = \frac{1}{2} \sum_{k=0}^{n} \Delta a_k + \sum_{k=1}^{n} \left( \sum_{j=k}^{n} \Delta a_j \right) \cos kx
\]
\[
= \sum_{k=0}^{n} D_k(x) \Delta a_k.
\]

Now
\[
\int_{0}^{\pi} |f(x) - f_n(x)| \, dx = \int_{0}^{\pi} \left| \sum_{k=n+1}^{\infty} D_k(x) \Delta a_k \right| \, dx
\]
\[
= \lim_{N \to \infty} \int_{0}^{\pi} \left| \sum_{k=n+1}^{N} D_k(x) \Delta a_k \right| \, dx = \lim_{N \to \infty} \int_{0}^{\pi} \left| \sum_{k=n+1}^{N} A_k D_k(x) \frac{\Delta a_k}{A_k} \right| \, dx
\]
\[
= \lim_{N \to \infty} \int_{0}^{\pi} \left| \sum_{k=n+1}^{N-1} \Delta A_k \sum_{\mu=1}^{k} D_\mu(x) \frac{\Delta a_\mu}{A_\mu} \right| \, dx
\]
\[
+ A_N \sum_{k=n+1}^{N} D_k(x) \frac{\Delta a_k}{A_k} - A_{n+1} \sum_{k=n+1}^{n} D_k(x) \frac{\Delta a_k}{A_k} \right| \, dx
\]
\[
< \lim_{N \to \infty} \left[ \sum_{k=n+1}^{N-1} |\Delta A_k| \int_{0}^{\pi} \left| \sum_{\mu=1}^{k} \frac{\Delta a_\mu}{A_\mu} D_\mu(x) \right| \, dx
\]
\[
+ A_N \int_{0}^{\pi} \left| \sum_{k=1}^{N} \frac{\Delta a_k}{A_k} D_k(x) \right| \, dx + A_{n+1} \int_{0}^{\pi} \left| \sum_{k=1}^{n} \frac{\Delta a_k}{A_k} D_k(x) \right| \, dx \right]
\]
\[
< \lim_{N \to \infty} \sum_{k=n+1}^{N-1} (k+1)|\Delta A_k| + A_N(N+1) + A_{n+1}(n+1), \]

by Lemma 1, since \(|\Delta a_k/A_k| < 1\) for each \(k\);
\[
< C \sum_{k=n+1}^{\infty} (k+1)|\Delta A_k| + (n+1)A_{n+1},
\]
since by Lemma 2

\[(N + 1)A_N = 0(1), \quad \text{as } N \to \infty.
\]

But by Lemma 3, \(\sum_{k=1}^{\infty} (k + 1)|\Delta A_k|\) converges, so by Lemmas 2 and 3 we have

\[
\lim_{n \to \infty} \int_0^{\pi} |f(x) - f_n(x)|dx = 0(1).
\]

This proves the theorem.

*Note.* This theorem gives a direct proof that the condition \(S'\) implies the necessary and sufficient condition of Garrett-Stanojevic [8, Theorem 1].

**References**


**Department of Mathematics, Kurukshetra University, Kurukshetra-132119, India**