

CONVERGENCE OF CERTAIN COSINE SUMS IN A METRIC SPACE - L

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ABSTRACT. In this paper a generalization of a theorem of Garrett and Stanojevic [6] has been obtained.

1. A sequence $\{a_n\}$ is said to be convex if $\Delta^2 a_n \geq 0$, where $\Delta^2 a_n = \Delta a_n - \Delta a_{n+1}$ and $\Delta a_n = a_n - a_{n+1}$. It is said to be quasi-convex if

$$\sum_{n=1}^{\infty} (n+1) |\Delta^2 a_n| < \infty.$$

It is well known that every bounded convex sequence is quasi-convex but the converse is not always true. The concept of null quasi-convex sequence has been further generalized by Sidon [1] in the following manner.

A sequence $\{a_n\}$ is said to satisfy condition S if

(i) $a_n = O(1)$, $n \rightarrow \infty$,

(ii) there exist numbers A_n such that $A_n \downarrow_0$ and $\sum A_n < \infty$,

(iii) $|\Delta a_n| \leq A_n$ for all n .

By taking $A_n = \sum_{m=n}^{\infty} |\Delta^2 a_m|$, we observe that every null quasi-convex sequence $\{a_n\}$ belongs to class S . As regards the converse, it is clear from the example $a_n = (-1)^n/n$, that a sequence $\{a_n\} \in S$ need not necessarily be quasi-convex.

A sequence of nonnegative numbers is said to be quasi-monotone if $a_{n+1} \leq a_n(1 + \alpha/n)$ for some constant $\alpha > 0$ and all $n > n_0(\alpha)$.

An equivalent definition is that $n^{-\beta} a_n \downarrow_0$ for some $\beta > 0$.

A sequence $\{a_n\}$ is said to satisfy condition S' if

(i) $a_n = O(1)$, $n \rightarrow \infty$,

(ii) there exist numbers A_n such that $\{A_n\}$ is a quasi-monotone sequence and $\sum_{n=1}^{\infty} A_n < \infty$,

(iii) $|\Delta a_n| \leq A_n$ for all n .

It is clear that the condition S' is weaker than the condition S .

2. Let

$$\frac{1}{2} a_0 + \sum_{n=1}^{\infty} a_n \cos nx \text{ be a trigonometric series.} \quad (2.1)$$

Received by the editors December 1, 1977.

AMS (MOS) subject classifications (1970). Primary 42A20, 42A32.

Key words and phrases. Integrability of cosine sums, metric space - L .

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Let

$$f(x) = \frac{1}{2} a_0 + \sum_{n=1}^{\infty} a_n \cos nx,$$

and

$$f_n(x) = \frac{1}{2} \sum_{k=0}^n \Delta a_k + \sum_{k=1}^n \left(\sum_{j=k}^n \Delta a_j \right) \cos kx.$$

Concerning the convergence of $f_n(x)$ in L -metric, Garrett and Stanojevic [6] proved the following theorem:

THEOREM A. *If $\{a_n\}$ is a null quasi-convex sequence, then $f_n(x)$ converges to $f(x)$ in L -metric.*

Babu Ram [9] and Maher M. H. Marzug [7] proved that the condition S also implies the conclusion of the Garrett-Stanojevic theorem.

The aim of this paper is to generalize this theorem and those of Babu Ram and Marzug by using condition S' in place of null quasi-convex sequence or condition S . We prove the following theorem.

THEOREM. *Let $a_n \in S'$, then $f_n(x)$ converges to $f(x)$ in L -metric.*

3. For the proof of this theorem we require the following lemmas:

LEMMA ([1], [3]). *If the numbers α_i , $i = 0, 1, \dots, k$, satisfy the condition $|\alpha_i| < 1$, then the following estimate is valid:*

$$\int_0^{\pi} \left| \sum_{i=0}^k \frac{\alpha_i \sin(i + 1/2)x}{2 \sin x/2} \right| dx \leq C(k + 1),$$

where C is an absolute constant.

LEMMA 2 [2]. *Let $\{a_n\}$ be a quasi-monotone sequence of constants. If $\sum_{n=1}^{\infty} a_n$ converges, then $na_n = o(1)$.*

LEMMA 3. *Let $\{a_n\}$ be a quasi-monotone sequence of constants. If $\sum_{n=1}^{\infty} a_n$ converges, then $\sum_{n=1}^{\infty} (n + 1)|\Delta a_n|$ converges.*

The proof is easy.

REMARK. This lemma also follows as a particular case of a more general theorem by Robertson [4].

THEOREM. *If the sequence $\{a_n\}$ is (ϕ, δ) monotone and $\sum a_n \Delta \phi_n$ converges, then the series $\sum \phi_{n+1} \Delta a_n$ is absolutely convergent.*

PROOF OF THE THEOREM. By Abel's transformation

$$\begin{aligned} f(x) &= \lim_{n \rightarrow \infty} \left[\frac{1}{2} a_0 + \sum_{k=1}^n a_k \cos kx \right] \\ &= \lim_{n \rightarrow \infty} \left[\frac{1}{2} a_0 + \sum_{k=1}^{n-1} D_k(x) \Delta a_k + a_n D_n(x) - \frac{1}{2} a_0 \right] \\ &= \lim_{n \rightarrow \infty} \left[\sum_{k=0}^{n-1} D_k(x) \Delta a_k + a_n D_n(x) \right] \\ &= \sum_{k=0}^{\infty} D_k(x) \Delta a_k, \quad \text{since } \lim_{n \rightarrow \infty} a_n D_n(x) = 0 \quad \text{if } x \neq 0, \end{aligned}$$

where $D_n(x) = 1/2 + \cos x + \cos 2x + \dots + \cos nx$.

Again, by Abel's transformation,

$$\begin{aligned} f_n(x) &= \frac{1}{2} \sum_{k=0}^n \Delta a_k + \sum_{k=1}^n \left(\sum_{j=k}^n \Delta a_j \right) \cos kx \\ &= \sum_{k=0}^n D_k(x) \Delta a_k. \end{aligned}$$

Now

$$\begin{aligned} \int_0^\pi |f(x) - f_n(x)| dx &= \int_0^\pi \left| \sum_{k=n+1}^{\infty} D_k(x) \Delta a_k \right| dx \\ &= \lim_{N \rightarrow \infty} \int_0^\pi \left| \sum_{k=n+1}^N D_k(x) \Delta a_k \right| dx = \lim_{N \rightarrow \infty} \int_0^\pi \left| \sum_{k=n+1}^N A_k D_k(x) \frac{\Delta a_k}{A_k} \right| dx \\ &= \lim_{N \rightarrow \infty} \int_0^\pi \left| \sum_{k=n+1}^{N-1} \Delta A_k \sum_{\mu=1}^k D_\mu(x) \frac{\Delta a_\mu}{A_\mu} \right. \\ &\quad \left. + A_N \sum_{k=1}^N D_k(x) \frac{\Delta a_k}{A_k} - A_{n+1} \sum_{k=1}^n D_k(x) \frac{\Delta a_k}{A_k} \right| dx \\ &\leq \lim_{N \rightarrow \infty} \left[\sum_{k=n+1}^{N-1} |\Delta A_k| \int_0^\pi \left| \sum_{\mu=1}^k \frac{\Delta a_\mu}{A_\mu} D_\mu(x) \right| dx \right. \\ &\quad \left. + A_N \int_0^\pi \left| \sum_{k=1}^N \frac{\Delta a_k}{A_k} D_k(x) \right| dx + A_{n+1} \int_0^\pi \left| \sum_{k=1}^n \frac{\Delta a_k}{A_k} D_k(x) \right| dx \right] \\ &\leq \lim_{N \rightarrow \infty} C \left[\sum_{k=n+1}^{N-1} (k+1) |\Delta A_k| + A_N (N+1) + A_{n+1} (n+1) \right], \end{aligned}$$

by Lemma 1, since $|\Delta a_k/A_k| \leq 1$ for each k ;

$$\leq C \sum_{k=n+1}^{\infty} (k+1) |\Delta A_k| + (n+1) A_{n+1},$$

since by Lemma 2

$$(N + 1)A_N = 0(1), \text{ as } N \rightarrow \infty.$$

But by Lemma 3, $\sum_{k=1}^{\infty} (k + 1)|\Delta A_k|$ converges, so by Lemmas 2 and 3 we have

$$\lim_{n \rightarrow \infty} \int_0^{\pi} |f(x) - f_n(x)| dx = 0(1).$$

This proves the theorem.

Note. This theorem gives a direct proof that the condition S' implies the necessary and sufficient condition of Garrett-Stanojevic [8, Theorem 1].

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