PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 72, Number 1, October 1978

# ANTIPODAL MANIFOLDS IN COMPACT SYMMETRIC SPACES OF RANK ONE

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ABSTRACT. Let $M$ be a compact Riemannian globally symmetric space of rank one. A theorem due to Helgason states that the antipodal manifold $A_x$ of a point $x \in M$ is again a symmetric space of rank one. We compute the multiplicities of the restricted roots of $A_x$ from those of $M$, obtaining a very convenient way to determine $A_x$.

**1. Introduction.** Let $M$ be a compact Riemannian globally symmetric space of rank one. Let $L$ denote the diameter of $M$ and if $x \in M$ let $A_x$ denote the corresponding *antipodal manifold*, that is the set of points $y \in M$ at distance $L$ from $x$; $A_x$ is indeed a manifold and with the Riemannian structure induced by $M$, is a symmetric space of rank one totally geodesic in $M$ (see Theorem 2.5). Associated to $M$ there is a triple of numbers $(p, q, \lambda)$ (see below for the definition) which appear in different places, for example in the expressions, in geodesic polar coordinates, of the Riemannian measure and the Laplace-Beltrami operator of $M$. Sometimes doing analysis on these spaces it is necessary to know $p_1$, $q_1$ and $\lambda_1$, the values of $p$, $q$ and $\lambda$ for an antipodal manifold of $M$ (see [**2**] and [**4**]). The complete list of compact Riemannian globally symmetric spaces of rank one, with the corresponding values of $p$, $q$ and $\lambda$ is (see [**4**, p. 171]):

The spheres $\mathbf{S}^n$, $n = 1, 2, \ldots : p = 0, q = n - 1, \lambda = \pi/2L$,

The real projective spaces $\mathbf{P}^n(\mathbf{R})$, $n = 2, 3, \ldots : p = 0, q = n - 1, \lambda = \pi/4L$,

The complex projective spaces $\mathbf{P}^n(\mathbf{C})$, $n = 4, 6, \ldots : p = n - 2, q = 1, \lambda = \pi/2L$,

The quaternion projective spaces $\mathbf{P}^n(\mathbf{H})$, $n = 8, 12, \ldots : p = n - 4, q = 3, \lambda = \pi/2L$,

The Cayley projective space $\mathbf{P}^{16}(\mathbf{Cay})$: $p = 8, p = 7, \lambda = \pi/2L$.

The superscripts denote the real dimension. The corresponding antipodal manifolds are also known ([**1**, pp. 437–467], [**5**, pp. 35 and 52]), but the computations involved are not simple. In this paper we compute $p_1$, $q_1$ and $\lambda_1$ directly from $p$, $q$ and $\lambda$. Since the triple $(p, q, \lambda)$ characterizes $M$, we also obtain a very convenient way to determine the antipodal manifolds of $M$.

Received by the editors October 20, 1977.

*AMS (MOS) subject classifications* (1970). Primary 53C35.

*Key words and phrases*. Geodesic symmetry, semisimple Lie group, involutive automorphism, restricted root, projective spaces.

**2.** We assume $\dim M > 1$. This is the case which interests us, and it has the convenient implication that the group $I(M)$ of isometries of $M$, in the compact open topology, is semisimple. Let $o$ be a fixed point in $M$ and $s_o$ the geodesic symmetry of $M$ with respect to $o$. Let $U$ denote the identity component of $I(M)$, $\mathfrak{u}$ the Lie algebra of $U$ and $\mathfrak{u} = \mathfrak{k} + \mathfrak{p}$ the decomposition of $\mathfrak{u}$ into eigenspaces of the involutive automorphism $d\gamma$ of $\mathfrak{u}$ which corresponds to the automorphism $\gamma$: $u \mapsto s_o u s_o$ of $U$. Here $\mathfrak{k}$ is the Lie algebra of the subgroup $K$ of $U$ which leaves $o$ fixed. Changing the distance function $d$ on $M$ by a constant factor we may assume that the differential of the mapping $\pi$: $u \mapsto u \cdot o$ of $U$ onto $M$ gives an isometry of $\mathfrak{p}$ (with the metric of the negative of the Killing form of $u$) onto $M_o$, the tangent space to $M$ at $o$.

Let $X \mapsto \operatorname{ad}(X)$ denote the adjoint representation of $\mathfrak{u}$. Select a vector $H \in \mathfrak{p}$ of length $L$. The space $\mathfrak{a} = \mathbf{R}H$ is a Cartan subalgebra of the symmetric space $M$ and we can select a positive restricted root $\alpha$ of $M$ such that $\frac{1}{2}\alpha$ is the only other possible positive restricted root. This means that the eigenvalues of $(\operatorname{ad} H)^2$ are 0, $\alpha(H)$ and possibly $(\frac{1}{2}\alpha(H))^2$; $\alpha(H)$ is purely imaginary. Let $\mathfrak{u} = \mathfrak{u}_0 + \mathfrak{u}_\alpha + \mathfrak{u}_{\alpha/2}$ be the corresponding decomposition of $\mathfrak{u}$ into eigenspaces and $\mathfrak{k}_\beta = \mathfrak{u}_\beta \cap \mathfrak{k}$, $\mathfrak{p}_\beta = \mathfrak{u}_\beta \cap \mathfrak{p}$ for $\beta = 0,\ \alpha,\ \frac{1}{2}\alpha$. Then $\mathfrak{p}_0 = \mathfrak{a}$ and $\mathfrak{k}_\beta = \operatorname{ad} H(\mathfrak{p}_\beta)$ for $\beta \neq 0$. We let $p = \dim \mathfrak{p}_{\alpha/2}$, $q = \dim \mathfrak{p}_\alpha$ and $\lambda = |\alpha(H)|/2L$.

The geodesics in $M$ are all closed and have length $2L$ and the exponential mapping Exp at $o$ is a diffeomorphism of the open ball in $M$ of center 0 and radius $L$ onto the complement $M - A_o$ (see [3, Chapter IX, §5]).

Lemma 2.1. (i) $\alpha(H) = \pm \pi i$ *if* $\frac{1}{2}\alpha$ *is a restricted root*; (ii) $\alpha(H) = \pm \pi i/2$ *if $H$ is not conjugate to* 0; (iii) $\alpha(H) = \pm \pi i$ *if* $\frac{1}{2}\alpha$ *is not a restricted root and $H$ is conjugate to* 0.

Proof. Considering Exp as a map of $\mathfrak{p}$ onto $M$ we have that $\operatorname{Exp} X = o$ for all $X \in \mathfrak{p}$ of length $2L$. Hence $d\operatorname{Exp}_{2H}$ vanishes identically on the orthogonal complement of $\mathfrak{a}$ in $\mathfrak{p}$. This orthogonal complement is precisely $\mathfrak{p}_\alpha + \mathfrak{p}_{\alpha/2}$. Using the formula for $d\operatorname{Exp}_{2H}$ [3, Theorem 4.1] it follows that $\alpha(2H) \in \pi i\mathbf{Z}$, and $\frac{1}{2}\alpha(2H) \in \pi i \mathbf{Z}$ when $\frac{1}{2}\alpha$ is a restricted root. Thus if $\frac{1}{2}\alpha$ is a restricted root $\alpha(H) = n\pi i$. We also have $|n| = 1$ because otherwise $n^{-1}H$ would be conjugate to 0 and of length less than $L$, which is impossible. This proves (i). To prove (ii) we observe that $\alpha(2n^{-1}H) = \pi i$ for some $n \in \mathbf{Z}$. As before it follows that $|n| \leqslant 2$. Now the assumption that $H$ is not conjugate to 0 implies $|n| = 1$. For (iii) we also have $|n| \leqslant 2$. But now $|n| = 2$ because $H$ is conjugate to 0. Q.E.D.

It will be convenient to choose $H$ so that the above lemma holds with the minus sign. Let $o_1$ denote the point $\operatorname{Exp}(-H)$.

Proposition 2.2. *Let $G$ denote the subgroup of $U$ leaving the point $o_1 \in M$ fixed. Then $\gamma(G) = G$.*

PROOF. We have $G = \exp(-H)K \exp H$ because $\exp(-H)\cdot o = o_1$. If $g = \exp(-H)k \exp H$, $k \in K$, then $\gamma(g) = \exp Hk \exp(-H)$. Hence to prove the assertion it suffices to show that $\exp H \cdot o = o_1$. We have $\operatorname{Exp}(t+2)H = \operatorname{Exp} tH$ for all $t \in \mathbf{R}$, therefore

$$\exp H \cdot o = \operatorname{Exp} H = \operatorname{Exp}(-H) = o_1. \qquad \text{Q.E.D.}$$

COROLLARY. 2.3. *Let $U^1$ denote the identity component of G and $K^1 = U^1 \cap K$. Then $(U^1, K^1)$ is a Riemannian symmetric pair.*

PROOF. It is a enough to check that $(U^1_\gamma)_0 \subset K^1 \subset U^1_\gamma$, where $U^1_\gamma$ is the set of fixed points of $\gamma$ in $U^1$ and $(U^1_\gamma)_0$ is the identity component of $U^1_\gamma$. If $u \in (U^1_\gamma)_0$ then $u \in (U_\gamma)_0$, hence $u \in K$ because $(U, K)$ is a symmetric pai [3, Theorem 3.3]. The other inclusion follows similarly. Q.E.D.

Let $\mathfrak{u}^1$ denote the Lie algebra of $U^1$ and let $\mathfrak{u}^1 = \mathfrak{k}^1 + \mathfrak{p}^1$ be the decomposition of $\mathfrak{u}^1$ into eigenspaces of the involutive automorphism of $\mathfrak{u}^1$ which corresponds to the automorphism $\gamma\colon U^1 \to U^1$.

PROPOSITION 2.4. (i) *When $\frac{1}{2}\alpha$ is a restricted root we have*

$$\mathfrak{k}^1 = \mathfrak{k}_0 + \mathfrak{k}_\alpha, \qquad \mathfrak{p}^1 = \mathfrak{p}_{\alpha/2}.$$

(ii) *If H is not conjugate to* 0

$$\mathfrak{k}^1 = \mathfrak{k}_0, \qquad \mathfrak{p}^1 = \mathfrak{p}_\alpha.$$

(iii) *If $\frac{1}{2}\alpha$ is not a restricted root and H is conjugate to* 0, *then*

$$\mathfrak{k}^1 = \mathfrak{k}_0 + \mathfrak{k}_\alpha, \qquad \mathfrak{p}^1 = \{0\}.$$

PROOF. The Lie algebra of $U^1$ is given by

$$\mathfrak{u}^1 = \{X \in \mathfrak{u}\colon \exp(tX)o_1 = o_1 \text{ for all } t \in \mathbf{R}\}.$$

Therefore $X \in \mathfrak{u}^1$ if and only if $\exp(tX)\exp(-H)o = \exp(-H)o$ for all $t \in \mathbf{R}$, which is equivalent to

$$\exp(\operatorname{Ad}(\exp H)\, tX) \in K \quad \text{all } t \in \mathbf{R},$$

where Ad denotes the adjoint representation of $U$. Hence

$$\mathfrak{u}^1 = \{X \in \mathfrak{u}\colon \operatorname{Ad}(\exp H)X \in \mathfrak{k}\}. \tag{2.1}$$

Let $X_\beta$ be a vector in $\mathfrak{u}_\beta$ for $\beta = 0, \alpha, \frac{1}{2}\alpha$. A direct computation which is left to the reader, yields

$$\operatorname{Ad}(\exp H)(X_0 + X_\alpha + X_{\alpha/2}) = X_0 + \cosh(\alpha(H))X_\alpha + \cosh\left(\tfrac{1}{2}\alpha(H)\right)X_{\alpha/2}$$
$$+\alpha(H)^{-1}\operatorname{ad} H\left(\sinh(\alpha(H))X_\alpha + 2\sinh\left(\tfrac{1}{2}\alpha(H)\right)X_{\alpha/2}\right).$$

Now the proposition follows from (2.1) and Lemma 2.1 by simple inspection. Q.E.D.

THEOREM 2.5. (*Cf.* [4] *Proposition* 5.1.) *When $\frac{1}{2}\alpha$ is a restricted root or H is not conjugate to* 0 *the orbit $M_1 = U^1 \cdot o$, with the Riemannian structure induced* *by $M$, is a symmetric space of rank one and a totally geodesic submanifold of $M$.*

*When $\frac{1}{2}\alpha$ is not a restricted root and $H$ is conjugate to* 0 *the orbit $M_1 = U^1 \cdot o$ reduces to a point $\{o\}$. In both cases $M_1 = A_{o_1}$.*

PROOF. Let $y \in M_1$. Writing $y = u \cdot o$, $u \in U^1$, we have $s_y(v \cdot o) = u\gamma(u^{-1}v) \cdot o \in M_1$ for $v \in U^1$ (see Proposition 2.2); hence $s_y(M_1) = M_1$. If $\gamma_y$ denotes the restriction of $s_y$ to $M_1$, then $\gamma_y$ is an involutive isometry of $M_1$ with $y$ as isolated fixed point. Thus $M_1$ is globally symmetric and $\gamma_y$ is the geodesic symmetry with respect to $y$.

To prove that $M_1$ is totally geodesic it suffices to show that each $M$-geodesic which is tangent to $M_1$ at $o$ is a path in $M_1$. Let $t \to \operatorname{Exp} tX$, $t \in \mathbf{R}$, be one of such geodesics. Then $X \in \mathfrak{p}^1 = \mathfrak{p} \cap \mathfrak{u}^1$. Hence $\operatorname{Exp} tX = \exp(tX) \cdot o \in M_1$, $t \in \mathbf{R}$, and our geodesic is a path in $M$. Consequently, the first two assertions follow from the definition of rank and Proposition 2.4.

Clearly, $M_1 \subset A_{o_1}$, both submanifolds are connected, and of the same dimension since $A_o = K \cdot o_1$. Hence $M_1 = A_{o_1}$. This completes the proof of the theorem. Q.E.D.

PROPOSITION 2.6. *Let $N$ be the kernel of the restriction homomorphism $U^1 \to I(M_1)$. Then $U^1/N$ is naturally isomorphic to the identity component $I_0(M_1)$ of $I(M_1)$.*

PROOF. If $\dim M_1 \leqslant 1$ then $M_1$ is either a point or $S^1$ (circle) and $\dim I_0(M_1) \leqslant 1$. The proposition is obviously true in these cases. Therefore we now assume $\dim M_1 > 1$.

Let $\mathfrak{z}$ be the center of $\mathfrak{u}^1$, then $\mathfrak{z} = (\mathfrak{z} \cap \mathfrak{k}^1) + (\mathfrak{z} \cap \mathfrak{p}^1)$. But $\mathfrak{z} \cap \mathfrak{p}^1 = \{0\}$ because $M_1$ is a symmetric space of rank one totally geodesic in $M$. Thus $\mathfrak{k}^1 \supset \mathfrak{z}$ and $K^1 \supset Z_0$, the identity component of the center $Z$ of $U^1$. Now $Z_0 \subset Z \subset K^1 \subset N$. Hence $\mathfrak{z} \subset \mathfrak{n}$, the Lie algebra of $N$. Therefore we have a surjective Lie algebra homomorphism $\mathfrak{u}^1/\mathfrak{z} \to \mathfrak{u}^1/\mathfrak{n}$, which implies thar $U^1/N$ is semisimple since $\mathfrak{u}^1/\mathfrak{z}$ is semisimple.

We also have $N \subset K^1$, and we want to consider the pair $(U^1/N, K^1/N)$. Given $u \in N$ and $v \in U^1$ we have $\gamma(u)(v \cdot o) = s_o u(\gamma(v) \cdot o) = s_o\gamma(v) \cdot o = v \cdot o$ (see Proposition 2.2), hence $\gamma(N) \subset N$. Therefore $\gamma$ induces an involutive analytic automorphism, denoted also by $\gamma$, of $U^1/N$. We shall prove that $\gamma$ turns $(U^1/N, K^1/N)$ into a symmetric pair. In fact, $K^1/N \subset U^1_\gamma/N \subset (U^1/N)_\gamma$. On the other hand, if $\exp(tX)N \in (U^1/N)_\gamma$, $X \in \mathfrak{u}^1$, then $\exp(-tX)\gamma(\exp tX) \in N$. Thus $-X + d\gamma(X) \in \mathfrak{n} \subset \mathfrak{k}^1$, which implies $X \in \mathfrak{k}^1$. Therefore $((U^1/N)_\gamma)_0 \subset K^1/N$.

At this point the proposition follows as a consequence of [**3**, Theorem 4.1]. Q.E.D.

When $I_0(M_1)$ is identified with $U^1/N$ the isotropy subgroup of $I_0(M_1)$ at $o \in M_1$ becomes $K^1/N$ and the decomposition of the Lie algebra of $I_0(M_1)$ under the corresponding involutive automorphism can be written as $\mathfrak{u}^1/\mathfrak{n} = \mathfrak{k}^1/\mathfrak{n} + \mathfrak{p}^1$. Therefore the constants $p_1$, $q_1$ and $\lambda_1$ associated to $M_1$ can be computed from $(\operatorname{ad} H_1)^2\colon \mathfrak{p}^1 \to \mathfrak{p}^1$, where $H_1 \in \mathfrak{p}^1$ is a vector of length $L$ ($L$ is also the diameter of $M_1$, see Theorem 2.5). In particular we obtain $\lambda_1 = \lambda$.

PROPOSITION 2.7. *If $H$ is not conjugate to* $0, p_1 = 0$ *and* $q_1 = q - 1$.

PROOF. Since $H$ and $H_1$ are conjugate under $K$, $(\operatorname{ad} H)^2$ and $(\operatorname{ad} H_1)^2$, as linear transformations of $\mathfrak{p}$, have the same eigenvalues with the same multiplicities. These eigenvalues are 0 and $\alpha(H)^2$ (see Lemma 2.1) with multiplicities 1 and $q$, respectively. Now the proposition follows because 0 is an eigenvalue of $(\operatorname{ad} H_1)^2\colon \mathfrak{p}^1 \to \mathfrak{p}^1$ and $\dim \mathfrak{p}^1 = \dim \mathfrak{p}_\alpha = q$ (see Proposition 2.4). Q.E.D.

From now on we shall assume that $\frac{1}{2}\alpha$ is a restricted root.

Let $\mathfrak{u}_{\mathbf{C}}$ be the complexification of $\mathfrak{u}$, $\theta$ the corresponding extension of $d\gamma$ and $B$ the Killing form of $\mathfrak{u}_{\mathbf{C}}$. Let $\mathfrak{h}$ be any maximal abelian subalgebra of $\mathfrak{u}$ containing $\mathfrak{a}$ and let $\mathfrak{h}_{\mathbf{C}}$, $\mathfrak{a}_{\mathbf{C}}$ and $\mathfrak{p}_{\mathbf{C}}$ denote the subspaces of $\mathfrak{u}_{\mathbf{C}}$ generated by $\mathfrak{h}$, $\mathfrak{a}$ and $\mathfrak{p}$, respectively. Then $\mathfrak{h}_{\mathbf{C}}$ is a Cartan subalgebra of $\mathfrak{u}_{\mathbf{C}}$. Now select compatible orderings in the dual spaces of $i\mathfrak{a}$ and $i\mathfrak{h}$, respectively. Let $\Delta$ denote the set of all nonzero roots (of $\mathfrak{u}_{\mathbf{C}}$ with respect to $\mathfrak{h}_{\mathbf{C}}$) and let $\Delta^+$ denote the set of all positive roots. Now for each $\lambda \in \Delta$ the linear function $\lambda^\theta$ defined by $\lambda^\theta(X) = \lambda(\theta X)$, $X \in \mathfrak{h}_{\mathbf{C}}$, is again a member of $\Delta$ and $\lambda(H)$ is either 0, or $\pm\frac{1}{2}\alpha(H)$, or $\pm\alpha(H)$ when $H \in \mathfrak{a}$. We may assume that $\lambda(H)$ is equal to $\beta(H)$ ($\beta = 0$, $\alpha$ or $\frac{1}{2}\alpha$) whenever $\lambda \in \Delta^+$, $H \in \mathfrak{a}$ being the vector already chosen.

LEMMA 2.8. *For each $\lambda \in \Delta^+$ such that $\lambda(H) = \beta(H)$, $\beta \neq 0$, select a nonzero vector $X_\lambda$ in the corresponding root subspace. Then the $X_\lambda - \theta X_\lambda$'s form a basis of* $\mathfrak{p}_\beta + i\mathfrak{p}_\beta$.

PROOF. Clearly the vectors $X_\lambda$'s, $\tau X_\lambda$'s form a basis of $\mathfrak{u}_\beta + i\mathfrak{u}_\beta$, $\tau$ being the conjugation of $\mathfrak{u}_{\mathbf{C}}$ with respect to $\mathfrak{u}$. When $\lambda$ runs over the set of all positive roots such that $\lambda(H) = \beta(H)$, $\beta \neq 0$, $\tau X_\lambda$ runs over all root subspaces corresponding to all negative roots $\mu$ such that $\mu(H) = -\beta(H)$. But this is precisely what happens with the vectors $\theta X_\lambda$. Thus the vectors $X_\lambda$'s and $\theta X_\lambda$'s form also a basis of $\mathfrak{u}_\beta + i\mathfrak{u}_\beta$. Now the assertion is clear. Q.E.D.

LEMMA 2.9. *For all $Y \in \mathfrak{p}_\beta + i\mathfrak{p}_\beta$, $\beta \neq 0$, we have that* $(\operatorname{ad} Y)^2\mathfrak{a}_{\mathbf{C}} \subset \mathfrak{a}_{\mathbf{C}}$.

PROOF. Let $\lambda, \mu \in \Delta^+$ such that $\lambda(H) = \mu(H)$. By the preceding lemma it suffices to prove that

$$\big(\operatorname{ad}(X_\lambda - \theta X_\lambda)\operatorname{ad}(X_\mu - \theta X_\mu) + \operatorname{ad}(X_\mu - \theta X_\mu)\operatorname{ad}(X_\lambda - \theta X_\lambda)\big)H \in \mathfrak{a}_{\mathbf{C}}.$$

We have

$$[X_\mu - \theta X_\mu, H] = -\mu(H)X_\mu + \mu^\theta(H)\theta X_\mu = -\mu(H)(X_\mu + \theta X_\mu),$$

and

$$\begin{aligned}\operatorname{ad}(X_\lambda - \theta X_\lambda)\operatorname{ad}(X_\mu - \theta X_\mu)H &= -\mu(H)[X_\lambda - \theta X_\lambda, X_\mu + \theta X_\mu] \\ &= -\mu(H)([X_\lambda, X_\mu] + [X_\lambda, \theta X_\mu] - [\theta X_\lambda, X_\mu] - \theta[X_\lambda, X_\mu]).\end{aligned}$$

Interchanging $\lambda$ and $\mu$ and adding we obtain

$$\begin{aligned}&(\operatorname{ad}(X_\lambda - \theta X_\lambda)\operatorname{ad}(X_\mu - \theta X_\mu) + \operatorname{ad}(X_\mu - \theta X_\mu)\operatorname{ad}(X_\lambda - \theta X_\lambda))H \\ &\qquad = -2\lambda(H)([X_\lambda, \theta X_\mu] - [\theta X_\lambda, X_\mu]).\end{aligned}$$

Now

$$[H, [X_\lambda, \theta X_\mu]] = (\lambda(H) + \mu^\theta(H))[X_\lambda, \theta X_\mu] = 0$$

and also $[H, [\theta X_\lambda, X_\mu]] = 0$. But $[X_\lambda, \theta X_\mu] - [\theta X_\lambda, X_\mu] \in \mathfrak{p}_{\mathbf{C}}$, hence it belongs to $\mathfrak{a}_{\mathbf{C}}$, in view of the maximality of $\mathfrak{a}_{\mathbf{C}}$. This proves our assertion and hence the lemma. Q.E.D.

Note that Lemma 2.9 holds even when $\operatorname{rank}(\mathfrak{u}) > 1$.

PROPOSITION 2.10. *Let* $q(Y)$ *be the complex number defined by* $(\operatorname{ad} Y)^2 H = q(Y)H$, $Y \in \mathfrak{p}_{\alpha/2} + i\mathfrak{p}_{\alpha/2}$. *Then* $q(Y) = (\pi/2L)^2\, B(Y, Y)$ *and* $(\operatorname{ad} Y)^2 Z = q(Y)Z$ *for all* $Z \in \mathfrak{a} + \mathfrak{p}_\alpha$.

PROOF. It is enough to consider $Y \in \mathfrak{p}_{\alpha/2}$. In fact, $q$ is a quadratic form on $\mathfrak{p}_{\alpha/2} + i\mathfrak{p}_{\alpha/2}$, since

$$(\operatorname{ad}(Y_1 + Y_2)^2 - \operatorname{ad}(Y_1)^2 - \operatorname{ad}(Y_2)^2)H = (\operatorname{ad} Y_1 \operatorname{ad} Y_2 + \operatorname{ad} Y_2 \operatorname{ad} Y_1)H.$$

The identity component $K_0^1$ of $K^1$ acts transitively on any sphere in $\mathfrak{a} + \mathfrak{p}_\alpha$ with center 0. In fact, $\mathfrak{a} + \mathfrak{p}_\alpha$ is orthogonal to $\mathfrak{p}_{\alpha/2}$ and therefore also stable under $K^1$. Moreover, the tangent space to the orbit $\operatorname{Ad}(K_0^1)H$ at the point $H$ is $[\mathfrak{k}^1, \mathfrak{a}]$ which equals $\mathfrak{p}_\alpha$ (see Proposition 2.4(i)). It follows that $\operatorname{Ad}(K_0^1)H$ is the sphere in $\mathfrak{a} + \mathfrak{p}_\alpha$ of radius $L$ and center 0.

Now take $x \in K_0^1$, $Y \in \mathfrak{p}_{\alpha/2}$, then

$$\operatorname{ad}(\operatorname{Ad}(x)Y)^2(\operatorname{Ad}(x)H) = \operatorname{Ad}(x)(\operatorname{ad} Y)^2 H = q(Y)\operatorname{Ad}(x)H.$$

This shows that $(\operatorname{ad} Y)^2$ acts as a scalar transformation on $\mathfrak{a} + \mathfrak{p}_\alpha$, multiplying by $q(Y)$.

If $Y$ has length $L$ the eigenvalues of $(\operatorname{ad} Y)^2$ as a linear transformation of $\mathfrak{p}$ are the same as those of $(\operatorname{ad} H)^2$. Thus $q(Y) = (\frac{1}{2}\alpha(H))^2$ since $\dim(\mathfrak{a} + \mathfrak{p}_\alpha) = q + 1$. From Lemma 2.1 we get $(\frac{1}{2}\alpha(H))^2 = -(\pi/2)^2$, from where the proposition follows by homogeneity. Q.E.D.

THEOREM 2.11. *If* $\frac{1}{2}\alpha$ *is a restricted root then* $p_1 = p - q - 1$ *and* $q_1 = q$.

PROOF. Given $H_1 \in \mathfrak{p}^1 = \mathfrak{p}_{\alpha/2}$ of length $L$ the eigenvalues of $(\operatorname{ad} H_1)^2$ in $\mathfrak{p}$ are 0, $-(\pi/2)^2$ and $-\pi^2$ with multiplicities 1, $p$ and $q$, respectively. Proposition 2.10 says that $(\operatorname{ad} H_1)^2$ on $\mathfrak{a} + \mathfrak{p}_\alpha$ has $-(\pi/2)^2$ as an eigenvalue of multiplicity $1 + q$. Therefore the eigenvalues of $(\operatorname{ad} H_1)^2$ in $\mathfrak{p}_{\alpha/2}$ are 0, $-(\pi/2)^2$ and $-\pi^2$ with multiplicities 1, $p - q - 1$ and $q$, respectively. Q.E.D.

COROLLARY 2.12. *The spheres* $\mathbf{S}^n$ $(n = 1, 2, \ldots)$, *the real projective spaces* $\mathbf{P}^n(\mathbf{R})$, $(n = 2, 3, \ldots)$, *the complex projective spaces* $\mathbf{P}^n(\mathbf{C})$ $(n = 4, 6, \ldots)$, *the quaternion projective spaces* $\mathbf{P}^n(\mathbf{H})$ $(n = 8, 12, \ldots)$ *and the Cayley projective plane* $\mathbf{P}^{16}(\mathbf{Cay})$ *are all the Riemannian globally symmetric spaces of rank* 1. *The superscripts denote the real dimension. The corresponding antipodal manifolds are in the respective cases: A point*, $\mathbf{P}^{n-1}(\mathbf{R})$, $\mathbf{P}^{n-2}(\mathbf{C})$, $\mathbf{P}^{n-4}(\mathbf{H})$, $\mathbf{S}^8$.

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