CURVATURE OF PRODUCT 3-MANIFOLDS

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Abstract. Let $M$ be a compact product 3-manifold without boundary. Let $g$ be a Riemannian metric on $M$. If $g$ has everywhere nonpositive sectional curvature, then $g$ is locally diffeomorphic to a product metric. The proof is by the method of pseudoframes.

1. Introduction. A. Preissmann [2] proved compact product manifolds do not admit Riemannian metrics with everywhere negative sectional curvature. Here we prove the following refinement for dimension three:

1.1 Theorem. Let $M$ be a compact product 3-manifold without boundary. Let $g$ be a Riemannian metric on $M$. If $g$ has everywhere nonpositive sectional curvature, then $g$ is locally diffeomorphic to a product metric.

Our proof will be by the method of pseudoframes. We begin with a brief exposition of this theory.

2. Pseudoframes. Let $g$ and $\bar{g}$ be Riemannian metrics on a smooth manifold $M$ of dimension $m$. At each point $x$ of $M$, we may find an automorphism $F$ of the tangent space $T_xM$ such that, for all $X$, $Y \in T_xM$, $g(X, Y) = \bar{g}(FX, FY)$. If $\omega^1, \ldots, \omega^m$ are a coframe at $x$, then we may write $g = g_{ij} \omega^i \otimes \omega^j$, and $\bar{g} = \bar{g}_{ij} \omega^i \otimes \omega^j$, where here and always we sum over repeated indices from 1 to $m$. If $F$ has matrix representation $F^i_j$ in this frame, then $g_{ij} = \bar{g}_{ij} F^i_j$. $F$ is determined in any case up to left translation by elements of the orthogonal group for $\bar{g}$. If we require that $F^i_j = F^j_i$ in frames orthonormal for $g$, and all eigenvalues of $F$ be positive, then $F$ is unique. The symmetric $F$ determined in this way at each point gives rise to a global tensor field of type $(1,1)$ which determines an automorphism of the tangent bundle. We use such an object to mimic the effect of a global change of frame. For this reason we call it a pseudoframe.

Remark. The symmetry condition on $F$ is important only to establish global existence of the tensor field. In what follows, we shall not assume $F$ to be symmetric.

Let $F(M)$ be the frame bundle of $M$, $p: F(M) \to M$ the natural projection. Given a standard basis of $\mathbb{R}^m$, we can consider each $u \in F(M)$ as a linear isomorphism $u: \mathbb{R}^m \to T_{p(u)}M$. Then the natural right action of $GL(m)$ on
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$F(\omega)$ is given by $R_u \omega = u\omega$ where $u\omega : R^n \to T_{p(u)}M$ is the composition
$R^n \overset{u}{\to} R^n \overset{u}{\to} T_{p(u)}M$ for $u \in GL(m)$.

**Definition.** A diffeomorphism $f : F(M) \to F(M)$ is a **bundle automorphism** if for all $u \in F(M)$, $p(f(u)) = p(u)$, and, for all $a \in GL(m)$, $f(u \cdot a) = (f(u)) \cdot a$.

If $g$ is a Riemannian metric on $M$, let $O(g)$ denote the subbundle of frames orthonormal for $g$.

Let $F$ be a pseudoframe on $M$ such that $g(X, Y) = \tilde{g}(FX, FY)$ for two Riemannian metrics $g$ and $\tilde{g}$. Define $f : F(M) \to F(M)$ by $f(u)(A) = F(u(A))$ for all $u \in F(M)$ and all $A \in R^n$. Then $p(f(u)) = p(u)$. For $a \in GL(m)$, $A \in R^n$, $u(a(A)) = u(a(A))$. Then $f(u \cdot a)(A) = F(u \cdot a(A))$. So $f(u \cdot a) = (f(u)) \cdot a$. A frame $u$ is in $O(g)$ if and only if $(A, B) = g(uA, uB)$ for all $A, B \in R^n$. But $g(uA, uB) = \tilde{g}(F(uA), F(uB)) = \tilde{g}(f(u)(A), f(u)(B))$. Thus $f$ is a bundle automorphism, and $f(O(g)) = O(\tilde{g})$.

Given a bundle automorphism $f$, we define an associated function $\tilde{F} : F(M) \to GL(m)$ by $\tilde{F}(u)(A) = u^{-1}(f(u)(A))$ for all $A \in R^n$. Then $f(u) = u\tilde{F}(u)$ and $\tilde{F}(uA) = a^{-1}F(u)a$. In matrix coordinates we have

$$\tilde{F}_j^i (ua) = a_s^{-1}\tilde{F}_j^i (ua) a_j^i.$$  \hspace{1cm} (2.1)

If $\phi$ is a connection on $F(M)$, we define the covariant derivative $D_\phi \tilde{F}_j^i$ by

$$\phi^i_j a^j_s D_\phi \tilde{F}_j^i = D_\phi \tilde{F}_j^i - \phi_j^i \tilde{F}_j^i + \tilde{F}_j^i \phi^i_j.$$ \hspace{1cm} (2.2)

For fixed $a \in GL(m)$, define $R_a : F(M) \to F(M)$ by $R_a(u) = u\cdot a$, and for fixed $u \in F(M)$, $L_a : GL(m) \to p^{-1}(p(u))$ by $L_a(a) = ua$. Then for $X \in T_u F(M)$,

$$f^*(X) = R_{\tilde{F}(u)^*}(X) + L_{\tilde{F}(u)^*}(\tilde{F}^{-1}(u) d\tilde{F}(X))$$ \hspace{1cm} (2.3)

where $\tilde{F}^{-1}(u) d\tilde{F}(x) \in T_idGL(m)$.

Now suppose that $f(O(g)) = O(\tilde{g})$ and let $\phi$ be a connection on $O(\tilde{g})$. Then $f^*\phi_j^i(X) = \phi_j^i(f^*X) = \phi_j^i(R_{f^*X} X) + \phi_j^i(L_{f^*X} \tilde{F}^{-1}(u) d\tilde{F}(X))$. But $R_u \phi_j^i = a_s^{-1} \phi_j^i a_j^i$. Thus

$$f^*\phi_j^i = \tilde{F}^{-1}(u) d\tilde{F}_j^i + \tilde{F}_j^i \phi^i_j.$$ \hspace{1cm} (2.4)

Let $\theta$ be the canonical horizontal $R^n$-valued one-form on $F(M)$, $\theta(X) = u^{-1}(p_u(X))$ for $X \in T_u F(M)$. Since $\theta$ vanishes on vectors tangent to the fiber $p^{-1}(x)$, $f^*\theta = R^x_{\tilde{F}} \theta$. If $\theta^i$ is the $i$th component of $\theta$ with respect to the standard basis of $R^n$, then $R^x_{\tilde{F}} \theta^i + \alpha_s^{-1} \theta^j$. Thus

$$f^*\theta^i = \tilde{F}^{-1}_i \theta^i.$$ \hspace{1cm} (2.5)

Note that these formulae are similar to those induced by a change of frame.

We may use $f$ to pull back a connection on $O(\tilde{g})$ to a connection on $O(g)$. For geometric purposes, we are most interested in what happens to the Levi-Civita connection under such an operation. Let $\omega$ and $\phi$ be the Levi-Civita connections on $O(g)$ and $O(\tilde{g})$ respectively. We define the **transition forms** $(f^*)$ by

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Note that each \((f^g)^j_k\) is a horizontal one-form. If \((f^g)^j_k = (f^g)^j_k f^*_k\), then \((f^g)^j_k\) is given explicitly by the formula

\[
2(f^g)^j_k = \left( D_\omega \tilde{F}_s^{-1} \right) \left( \tilde{E}_i \right) \left\{ \frac{f_{j\bar{k}}}{\left. \right|} \tilde{F}_s \tilde{F}_s^{-1} - \tilde{F}_k \tilde{F}_l \right\} - \left( D_\omega \tilde{F}_s^{-1} \right) \left( \tilde{E}_i \right) \left\{ \frac{f_{j\bar{k}}}{\left. \right|} \tilde{F}_s \tilde{F}_s^{-1} - \tilde{F}_k \tilde{F}_l \right\} \tag{2.7}
\]

where \(\tilde{E}_i \in T_u \mathcal{F}(M)\) such that \(p_u \tilde{E}_i = u(E_i), E_i \in \mathbb{R}^m\) the \(t\)th leg of the standard basis.

3. Proof of Theorem 1.1. Let \(N\) be a compact, oriented 2-manifold without boundary, and \(S^1\) the unit circle; \(g\) a product metric on \(N \times S^1\). Let \(R(g)\) be the subbundle of \(O(g)\) consisting of frames such that \(u(Ex), u(E_2)\) are an oriented basis of \(T_p(u)N\). Let \(\omega\) be the Levi-Civita connection on \(O(g)\).

If \(\pi: M \to N\) is the natural projection, let \(\tilde{g}\) be a Riemannian metric on \(M\) such that, for \(u \in R(g)\), \(\tilde{g}_{s3} = \tilde{g}(u(E_3), u(E_3)) = \pi^* \lambda\), for some positive function \(\lambda\) on \(N\).

3.1 Lemma. There exists a sequence of bundle automorphisms \(O(g) \to O(\tilde{g}) \to O(\tilde{g})\) such that for \(u \in R(g)\),

- \((A)\) \(\tilde{F}_{s} = \tilde{F}_{s} = \tilde{F}_{s} = \tilde{F}_{s} = 0\),
- \((B)\) \(\tilde{F}_{s} = 1\),

and for \(v \in R(\tilde{g}) = f(R(g))\)

- \((C)\) \(\tilde{H}_{s} = \tilde{H}_{s} = 0\),
- \((D)\) \(\tilde{H}_{s} = \tilde{H}_{s} = \tilde{H}_{s} = 0\),
- \((E)\) \(\tilde{H}_{s} = \tilde{H}_{s} = 0\).

Proof. If we require additionally that \(\tilde{F}_{s} = \tilde{F}_{s} = 0\), then \(f\) and \(h\) are determined uniquely.

Note that for \(u \in R(g)\), \(\tilde{g}(u(E_3), u(E_3)) = 1, \tilde{g}(u(E_3), u(E_3)) = \theta, i = 1, 2\). Also, \(\tilde{H}_{s} = \mu = (\pi^* \lambda)^{-1/2}\).

Let \(\phi(\psi)\) be the Levi-Civita connection on \(O(\tilde{g})\) \((O(\tilde{g}))\), \(\Phi(\Psi)\) its curvature form. If \(\Phi_j = \Phi_{ij} \theta^i \wedge \theta^j\) then the sectional curvature \(\tilde{c}_{ij}\) of the plane spanned by \(u(E_i), u(E_j)\), \(u \in O(\tilde{g})\) is given by \(\Phi_{ij}\).

Let \(\psi_j = \psi_{jk} h^* \theta^k\). Then by (2.7), on \(R(\tilde{g})\),

\[
\psi_{j3} = 0, \quad \psi_{j2} = \psi_{j2}^3, \tag{3.1}
\]

\[
(h^g)^{11}_{11} = (h^g)^{12}_{22} = 0, \quad (h^g)^{33}_{33} = - (h^g)^{32}_{22}. \tag{3.2}
\]

(Note that \(\tilde{g}_{j3} = \pi^* \lambda\) is required to prove (3.2).)

On \(R(\tilde{g})\), let \(dv_{\tilde{g}} = h^* \theta^1 \wedge h^* \theta^2 \wedge h^* \theta^3 \wedge \psi_{j2}^1, \psi_{j2}^1, dv_{\tilde{g}} = \theta^1 \wedge \theta^2 \wedge \theta^3 \wedge \psi_{j2}^1\).

3.2 Lemma.

\[
\int_{R(\tilde{g})} h^* \Phi_{j13}^1 dv_{\tilde{g}} = \int_{R(\tilde{g})} \mu \psi_{j13}^1 dv_{\tilde{g}} - \int_{R(\tilde{g})} (h^g)^{12}_{22}, (h^g)^{32}_{22}. \tag{3.3}
\]
Proof.

\[
\int_{R(\hat{g})} h^*\Phi^1_{313} dv_{\hat{g}} = -\int_{R(\hat{g})} h^*\Phi^1_3 \wedge h^*\theta^2 \wedge \psi^1_2
\]

\[
= -\int_{R(\hat{g})} d\psi^1_3 \wedge h^*\theta^2 \wedge \psi^1_2 + \int_{R(\hat{g})} d(h^*\theta)^1 \wedge h^*\theta^2 \wedge \psi^1_2
\]

\[
- \int_{R(\hat{g})} h^*\phi^1_2 \wedge h^*\phi^2_3 \wedge h^*\theta^2 \wedge \psi^1_2
\]

\[
= \int_{R(\hat{g})} \mu \Psi^1_{313} dv_{\hat{g}} + \int_{R(\hat{g})} (h^*\theta)^1 \wedge h^*\theta \wedge h^*\phi^2_3 \wedge \psi^1_2
\]

\[
+ \int_{R(\hat{g})} (h^*\theta)^1 \wedge h^*\phi^3_1 \wedge h^*\theta \wedge \psi^1_2
\]

Now, \( \int_{R(\hat{g})} h^*\Phi^1_{313} dv_{\hat{g}} = \int_{R(\hat{g})} h^*\Phi^2_{323} dv_{\hat{g}} \), employing the same derivation for \( \Phi^2_{323} \) we obtain the desired result by cancelling cross-terms using (3.1) and (3.2).

Note that by (3.2), the second term of (3.3) is nonnegative.

3.3 Lemma. \( 0 < \int_{R(\hat{g})} \mu \Psi^1_{313} dv_{\hat{g}} \). The equality holds only if \( \hat{g} \) is a product metric.

Proof. Let \( C \) be a simple closed curve on \( N \). Let \( \sigma: C \times S^1 \rightarrow R(g) \) be a section such that, at each \( x \in C \), \( \sigma(x)(E_1) \) is the oriented tangent vector to \( C \), and \( \sigma(x)(E_2) \) the outward normal. Now, near \( C \), we can choose \( f \) so, in addition to conditions (A) and (B), we have \( \tilde{F}^{-1} = \tilde{F}^{-12} = 0 \) on the image of \( \sigma \). Since \( \sigma^*\theta^2 = 0 \) and \( \sigma^*\theta^2 = \tilde{F}_{12}^{-1}\theta^j \), we have \( \sigma^*\theta^2 = 0 \). Thus the \( \hat{g} \) volume element on \( C \times S^1 \) is \( dv = \sigma^*f^* (\theta^1 \wedge \theta^2) \). Then

\[
\int_{C \times S^1} \sigma^*f^* \Psi^1_{313} dv = \int_{C \times S^1} \sigma^*f^* \Psi^1_3 = \int_{C \times S^1} (f\xi)^2_{13}(f\xi)^2_{31} - (f\xi)^2_{33}(f\xi)^2_{11} dv.
\]

(3.5)

Since \( d\sigma^*f^* = 0 \), \( (f\xi)^2_{31} = (f\xi)^2_{13} \), and by (2.7), \((f\xi)^2_{33} = 0 \). This must be true for every curve \( C \) on \( N \), and \( \mu \) is a function on \( N \) only. Thus we have the desired inequality. The equality holds only if \((f\xi)^2_{13} \) vanishes pointwise on \( R(g) \). By (2.7), this implies that \( \tilde{F}^{-1} D_{i} \tilde{F}^{-12}(E_3) = 0 \). Since we may replace \( g \) with any other product metric, we conclude that the \( \tilde{F}^i_j \), \( i, j = 1, 2 \), are functions on \( N \) only, so that \( \hat{g} \) is itself a product metric.

3.4 Proposition. If the sectional curvature of \( \hat{g} \) is nonpositive, \( \hat{g} \) is locally diffeomorphic to a product metric.

Proof. By Lemmas 3.2 and 3.3, we may assume that \( \hat{g} = g \). Then, by
Lemma 3.1, \((h^g)^3_{11} = (h^g)^3_{22} = (h^g)^3_{12} = (h^g)^3_{21} = 0\) on \(R(g)\). By (2.7), this also means that \((h^g)^3_{23} = 0\). Since \(\mu\) is a function on \(N\) only, we can alter the product metric \(g\) so that \(\tilde{F}^{-1}_{1} = \tilde{F}^{-1}_{2} = 1\). (The \((h^g)^1_{ij}\) and \((h^g)^3_{ij}\), \(i,j = 1, 2\), will remain equal to zero when this is done.) Then \((h^g)^1_{i} = 0\). Now, by Lemma 3.2, \(h^s\phi_{3i3}\) must vanish on \(R(g)\) if it is to be nonpositive. But this implies that \(d((h^g)^1_{3i}(E^3)) - ((h^g)^1_{3i})^2 = 0\). Thus \((h^g)^1_{3i} = (h^g)^1_{3i} = 0\), and \((h^g)^1_{i} = 0\) for all \(i,j\). It now follows from the De Rham Decomposition Theorem that, on each simply-connected open set \(U\) of \(M\), \(\tilde{g}\) is a product metric for some product structure on \(U\). Note that the product structure may differ from the original one induced by the inclusion \(i: U \rightarrow N \times S^1\) by a diffeomorphism.

We now remove the restriction that \(\tilde{g}_{33}\) be a function on \(N\) only.

3.5 Proposition. Let \(N\) be a surface, \(S^1\) the unit circle, \(\pi: N \times S^1 \rightarrow N\) the natural projection. Let \(t\) be a unit-length parameter on \(S^1\) (i.e., \(t = 0\) and \(t = 1\) are identified). Then, for any Riemannian metric \(g\) on \(N \times S^1\), there exists a diffeomorphism \(\phi: N \times S^1 \rightarrow N \times S^1\) such that \(\phi^*g(d/dt, d/dt) = \pi^*\lambda\), for some positive function \(\lambda\) on \(N\).

Proof. Let \((x_1, x_2, t)\) be a local product coordinate chart on \(N \times S^1\), and let \(\mu^{-2} = g(d/dt, d/dt)\). Define \(\phi\) by

\[
\begin{align*}
\phi_1(x_1, x_2, t) &= x_1, \\
\phi_2(x_1, x_2, t) &= x_2, \\
\phi_3(x_1, x_2, t) &= K(x_1, x_2) \int_0^t \mu(x_1, x_2, s) ds,
\end{align*}
\]

where \(K^{-1} = \int_0^1 \mu(x_1, x_2, t) dt\). It is clear that \(\phi\) is a diffeomorphism, and it is easy to calculate that \(\phi^*g_{33} = K^2\). But by construction, \(K = \pi^*\lambda\) for some positive function \(\lambda\) on \(N\). For compact, oriented 3-manifolds, Theorem 1.1 now follows from Propositions 3.4 and 3.5. If a product 3-manifold \(M\) is compact, but not oriented, we may apply our results to the orientation covering \(\tilde{M}\); the local diffeomorphism found there will project to a local diffeomorphism of \(M\).

References


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