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CURVATURE OF PRODUCT 3-MANIFOLDS

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ABSTRACT. Let M be a compact product 3-manifold without boundary. Let g be a Riemannian metric on M. If g has everywhere nonpositive sectional curvature, then g is locally diffeomorphic to a product metric. The proof is by the method of pseudoframes.

1. Introduction. A. Preissmann [**2**] proved compact product manifolds do not admit Riemannian metrics with everywhere negative sectional curvature. Here we prove the following refinement for dimension three:

1.1 THEOREM. *Let M be a compact product 3-manifold without boundary. Let g be a Riemannian metric on M. If g has everywhere nonpositive sectional curvature, then g is locally diffeomorphic to a product metric.*

Our proof will be by the method of pseudoframes. We begin with a brief exposition of this theory.

2. Pseudoframes. Let g and $\bar{g}$ be Riemannian metrics on a smooth manifold M of dimension m. At each point x of M, we may find an automorphism F of the tangent space T_xM such that, for all $X, Y \in T_xM$, $g(X, Y) = \bar{g}(FX, FY)$. If $\omega^1, \dots, \omega^m$ are a coframe at x, then we may write $g = g_{ij}\omega^i \otimes \omega^j$, and $\bar{g} = \bar{g}_{ij}\omega^i \otimes \omega^j$, where here and always we sum over repeated indices from 1 to m. If F has matrix representation F^i_j in this frame, then $g_{ij} = F^i_s\bar{g}_{st}F^t_j$. F is determined in any case up to left translation by elements of the orthogonal group for $\bar{g}$. If we require that $F^i_j = F^j_i$ in frames orthonormal for g, and all eigenvalues of F be positive, then F is unique. The symmetric F determined in this way at each point gives rise to a global tensor field of type (1,1) which determines an automorphism of the tangent bundle. We use such an object to mimic the effect of a global change of frame. For this reason we call it a *pseudoframe*.

REMARK. The symmetry condition on F is important only to establish global existence of the tensor field. In what follows, we shall not assume F to be symmetric.

Let $\mathbf{F}(M)$ be the frame bundle of M, p: $\mathbf{F}(M) \to M$ the natural projection. Given a standard basis of $\mathbf{R}^m$, we can consider each $u \in \mathbf{F}(M)$ as a linear isomorphism u: $\mathbf{R}^m \to T_{p(u)}M$. Then the natural right action of $GL(m)$ on

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$\mathbf{F}(M)$ is given by $R_a u = ua$ where ua: $\mathbf{R}^m \to T_{p(u)}$ is the composition $\mathbf{R}^m \xrightarrow{a} \mathbf{R}^m \xrightarrow{u} T_{p(u)}M$ for $a \in GL(m)$.

DEFINITION. A diffeomorphism f: $\mathbf{F}(M) \to \mathbf{F}(M)$ is a *bundle automorphism* if for all $u \in \mathbf{F}(M), p(f(u)) = p(u)$, and, for all $a \in GL(m), f(ua) = (f(u))a$.

If g is a Riemannian metric on M, let $O(g)$ denote the subbundle of frames orthonormal for g.

Let F be a pseudoframe on M such that $g(X, Y) = \bar{g}(FX, FY)$ for two Riemannian metrics g and $\bar{g}$. Define f: $\mathbf{F}(M) \to \mathbf{F}(M)$ by $f(u)(A) = F(u(A))$ for all $u \in \mathbf{F}(M)$ and all $A \in \mathbf{R}^m$. Then $p(f(u)) = p(u)$. For $a \in GL(m)$, $A \in \mathbf{R}^m$, $ua(A) = u(a(A))$. Then $f(ua)(A) = F(ua(A)) = f(u)a(A)$. So $f(ua) = (f(u))a$. A frame u is in $O(g)$ if and only if $(A, B) = g(uA, uB)$ for all $A, B \in \mathbf{R}^m$. But $g(uA, uB) = \bar{g}(F(uA), F(uB)) = \bar{g}(f(u)(A), f(u)(B))$. Thus f is a bundle automorphism, and $f(O(g)) = O(\bar{g})$.

Given a bundle automorphism f, we define an associated function $\bar{F}$: $\mathbf{F}(M) \to GL(m)$ by $\bar{F}(u)(A) = u^{-1}(f(u)(A))$ for all $A \in \mathbf{R}^m$. Then $f(u) = u\bar{F}(u)$ and $\bar{F}(ua) = a^{-1}\bar{F}(u)a$. In matrix coordinates we have

$$\bar{F}_j^i(ua) = a_s^{-1i}\bar{F}_t^s(u)a_j^t. \tag{2.1}$$

If ϕ is a connection on $\mathbf{F}(M)$, we define the covariant derivative $D_\phi \bar{F}_j^i$ by

$$d\bar{F}_j^i = D_\phi \bar{F}_j^i - \phi_s^i \bar{F}_t^s + \bar{F}_s^i \phi_t^s. \tag{2.2}$$

For fixed $a \in GL(m)$, define R_a: $\mathbf{F}(M) \to \mathbf{F}(M)$ by $R_a(u) = ua$, and for fixed $u \in \mathbf{F}(M)$, L_u: $GL(m) \to p^{-1}(p(u))$ by $L_u(a) = ua$. Then for $X \in T_u\mathbf{F}(M)$,

$$f_*(X) = R_{\bar{F}(U)*}(X) + L_{f(u)*}\big(\bar{F}^{-1}(u)d\bar{F}(X)\big) \tag{2.3}$$

where $\bar{F}^{-1}(u)d\bar{F}(x) \in T_{id}GL(m)$.

Now suppose that $f(O(g)) = O(\bar{g})$ and let ϕ be a connection on $O(\bar{g})$. Then $f^*\phi_j^i(X) = \phi_j^i(f_*X) = \phi_j^i(R_{\bar{F}*}X) + \phi_j^i(L_{f*}\bar{F}^{-1}d\bar{F}(X))$. But $R_{a*}\phi_j^i = a_s^{-1i}\phi_t^s a_j^t$. Thus

$$f^*\phi_j^i = \bar{F}_s^{-1i} d\bar{F}_j^s + \bar{F}_s^{-1i}\phi_t^s \bar{F}_j^t. \tag{2.4}$$

Let θ be the canonical horizontal R^m-valued one-form on $\mathbf{F}(M)$, $\theta(X) = u^{-1}(p_*(X))$ for $X \in T_u\mathbf{F}(M)$. Since θ vanishes on vectors tangent to the fiber $p^{-1}(x)$, $f^*\theta = R_{\bar{F}}^*\theta$. If θ^i is the ith component of θ with respect to the standard basis of R^m, then $R_a^*\theta^i + a_s^{-1i}\theta^s$. Thus

$$f^*\theta^i = \bar{F}_s^{-1i}\theta^s. \tag{2.5}$$

Note that these formulae are similar to those induced by a change of frame.

We may use f to pull back a connection on $O(\bar{g})$ to a connection on $O(g)$. For geometric purposes, we are most interested in what happens to the Levi-Civita connection under such an operation. Let ω and ϕ be the Levi-Civita connections on $O(g)$ and $O(\bar{g})$ respectively. We define the *transition forms* (f^g) by

$$(f^g)^i_j = \omega^i_j - f^*\phi^i_j. \tag{2.6}$$

Note that each $(f^g)^i_j$ is a horizontal one-form. If $(f^g)^i_j = (f^g)^i_{jk} f^*\theta^k$, then $(f^g)^i_{jk}$ is given explicitly by the formula

$$2(f^g)^i_{jk} = \left(D_\omega \bar{F}_s^{-1i}\left(\bar{E}_t\right)\right)\left(\bar{F}^s_j\bar{F}^t_k - \bar{F}^s_k\bar{F}^t_j\right) - \left(D_\omega \bar{F}_s^{-1j}\left(\bar{E}_t\right)\right)\left(\bar{F}^s_i\bar{F}^t_k - \bar{F}^s_k\bar{F}^t_i\right)$$
$$- \left(D_\omega \bar{F}_s^{-1k}\left(\bar{E}_t\right)\right)\left(\bar{F}^s_i\bar{F}^t_j - \bar{F}^s_j\bar{F}^t_i\right) \tag{2.7}$$

where $\bar{E}_t \in T_u\mathbf{F}(M)$ such that $p_*\bar{E}_t = u(E_t)$, $E_t \in \mathbf{R}^m$ the tth leg of the standard basis.

3. Proof of Theorem 1.1. Let N be a compact, oriented 2-manifold without boundary, and S^1 the unit circle; g a product metric on $N \times S^1$. Let $R(g)$ be the subbundle of $O(g)$ consisting of frames such that $u(E_1)$, $u(E_2)$ are an oriented basis of $T_{p(u)}N$. Let ω be the Levi-Civita connection on $O(g)$.

If $\pi\colon M \to N$ is the natural projection, let $\bar{g}$ be a Riemannian metric on M such that, for $u \in R(g)$, $\bar{g}_{33} = \bar{g}(u(E_3), u(E_3)) = \pi^*\lambda$, for some positive function λ on N.

3.1 Lemma. *There exists a sequence of bundle automorphisms $O(g) \xrightarrow{f} O(\hat{g}) \xrightarrow{h} O(\bar{g})$ such that for $u \in R(g)$,*

(A) $\bar{F}_3^{-11} = \bar{F}_3^{-12} = \bar{F}_1^{-13} = \bar{F}_2^{-13} = 0$,

(B) $\bar{F}_3^{-13} = 1$,

and for $v \in R(\hat{g}) = f(R(g))$

(C) $\bar{H}_3^{-11} = \bar{H}_3^{-12} = 0$,

(D) $\bar{H}_1^{-11} = \bar{H}_2^{-12} = \bar{H}_3^{-13}$,

(E) $\bar{H}_1^{-12} = \bar{H}_2^{-11} = 0$.

Proof. If we require additionally that $\bar{F}_2^{-11} = \bar{F}_1^{-12}$, then f and h are determined uniquely.

Note that for $u \in R(g)$, $\hat{g}(u(E_3), u(E_3)) = 1$, $\hat{g}(u(E_3), u(E_i)) = \theta$, $i = 1, 2$. Also, $\bar{H}_3^{-13} = \mu = (\pi^*\lambda)^{-1/2}$.

Let ϕ (ψ) be the Levi-Civita connection on $O(\bar{g})$ $(O(\hat{g}))$, Φ (Ψ) its curvature form. If $\Phi^i_j = \Phi^i_{jst}\theta^s \wedge \theta^t$ then the sectional curvature $\bar{\sigma}_{ij}$ of the plane spanned by $u(E_i)$ and $u(E_j)$, $u \in O(\bar{g})$ is given by Φ^i_{jij}.

Let $\psi^i_j = \psi^i_{jk}h^*\theta^k$. Then by (2.7), on $R(\hat{g})$,

$$\psi^i_{33} = 0, \quad \psi^3_{12} = \psi^3_{21}, \tag{3.1}$$

$$(h^g)^3_{11} = (h^g)^3_{22} = 0, \quad (h^g)^3_{12} = -(h^g)^3_{21}. \tag{3.2}$$

(Note that $\bar{g}_{33} = \pi^*\lambda$ is required to prove (3.2).)

On $R(\hat{g})$, let $dv_{\bar{g}} = h^*\theta^1 \wedge h^*\theta^2 \wedge h^*\theta^3 \wedge \psi^1_2$, $dv_{\hat{g}} = \theta^1 \wedge \theta^2 \wedge \theta^3 \wedge \psi^1_2$.

3.2 Lemma.

$$\int_{R(\hat{g})} h^*\Phi^1_{313}dv_{\bar{g}} = \int_{R(\hat{g})} \mu\Psi^1_{313}dv_{\hat{g}} - \int_{R(\hat{g})} \left(h^{\hat{g}}\right)^3_{12}\left(h^{\hat{g}}\right)^3_{21}dv_{\bar{g}}. \tag{3.3}$$

PROOF.

$$\begin{aligned}
\int_{R(\hat{g})} h^*\Phi^1_{313} dv_{\bar{g}} &= -\int_{R(\hat{g})} h^*\Phi^1_3 \wedge h^*\theta^2 \wedge \psi^1_2 \\
&= -\int_{R(\hat{g})} d\psi^1_3 \wedge h^*\theta^2 \wedge \psi^1_2 + \int_{R(\hat{g})} d\left(h^{\hat{g}}\right)^1_3 \wedge h^*\theta^2 \wedge \psi^1_2 \\
&\quad - \int_{R(\hat{g})} h^*\phi^1_2 \wedge h^*\phi^2_3 \wedge h^*\theta^2 \wedge \psi^1_2 \\
&= \int_{R(\hat{g})} \mu\Psi^1_{313} dv_{\hat{g}} + \int_{R(\hat{g})} \left(h^{\hat{g}}\right)^1_3 \wedge h^*\theta^1 \wedge h^*\phi^2_1 \wedge \psi^1_2 \\
&\quad + \int_{R(\hat{g})} \left(h^{\hat{g}}\right)^1_3 \wedge h^*\theta^3 \wedge h^*\phi^2_3 \wedge \psi^1_2 \\
&\quad + \int_{R(\hat{g})} \left(h^{\hat{g}}\right)^1_3 \wedge h^*\phi^2_3 \wedge h^*\theta^2 \wedge \psi^1_2.
\end{aligned}$$

Now, $\int_{R(\hat{g})} h^*\Phi^1_{313} dv_{\bar{g}} = \int_{R(\hat{g})} h^*\Phi^2_{323} dv_{\bar{g}}$, employing the same derivation for Φ^2_{323} we obtain the desired result by cancelling cross-terms using (3.1) and (3.2).

Note that by (3.2), the second term of (3.3) is nonnegative.

3.3 LEMMA. $0 \leqslant \int_{R(\hat{g})} \mu\Psi^1_{313} dv_{\hat{g}}$. *The equality holds only if $\hat{g}$ is a product metric.*

PROOF. Let C be a simple closed curve on N. Let $\sigma\colon C \times S^1 \to R(g)$ be a section such that, at each $x \in C$, $\sigma(x)(E_1)$ is the oriented tangent vector to C, and $\sigma(x)(E_2)$ the outward normal. Now, near C, we can choose f so, in addition to conditions (A) and (B), we have $\bar{F}_1^{-12} = \bar{F}_3^{-12} = 0$ on the image of σ. Since $\sigma^*\theta^2 = 0$, and $f^*\theta^2 = \bar{F}_j^{-12}\theta^j$, we have $\sigma^* f^*\theta^2 = 0$. Thus the $\hat{g}$ volume element on $C \times S^1$ is $dv = \sigma^* f^*(\theta^1 \wedge \theta^3)$. Then

$$\int_{C\times S^1} \sigma^* f^*\Psi^1_{313} dv = \int_{C\times S^1} \sigma^* f^*\Psi^1_3 = \int_{C\times S^1} (f^g)^2_{13}(f^g)^2_{31} - (f^g)^2_{33}(f^g)^2_{11}\, dv. \tag{3.5}$$

Since $d\sigma^* f^{*2} = 0$, $(f^g)^2_{31} = (f^g)^2_{13}$, and by (2.7), $(f^g)^2_{33} = 0$. This must be true for every curve C on N, and μ is a function on N only. Thus we have the desired inequality. The equality holds only if $(f^g)^2_{13}$ vanishes pointwise on $R(g)$. By (2.7), this implies that $\bar{F}_2^{-11} D_\omega \bar{F}_2^{-12}(\bar{E}_3) = 0$. Since we may replace g with any other product metric, we conclude that the $\bar{F}^i_j$, $i, j = 1, 2$, are functions on N only, so that $\hat{g}$ is itself a product metric.

3.4 PROPOSITION. *If the sectional curvature of $\bar{g}$ is nonpositive, $\bar{g}$ is locally diffeomorphic to a product metric.*

PROOF. By Lemmas 3.2 and 3.3, we may assume that $\hat{g} = g$. Then, by

Lemma 3.1, $(h^g)^3_{11} = (h^g)^3_{22} = (h^g)^3_{12} = (h^g)^3_{21} = 0$ on $R(g)$. By (2.7), this also means that $(h^g)^1_{23} = 0$. Since μ is a function on N only, we can alter the product metric g so that $\bar{F}_1^{-11} = \bar{F}_2^{-12} = 1$. (The $(h^g)^3_{ij}$ and $(h^g)^i_{j3}$, $i, j = 1, 2$, will remain equal to zero when this is done.) Then $(h^g)^1_2 = 0$. Now, by Lemma 3.2, $h^*\Phi^1_{313}$ must vanish on $R(g)$ if it is to be nonpositive. But this implies that $d(h^g)^1_{33}(E_3) - ((h^g)^1_{33})^2 = 0$. Thus $(h^g)^1_{33} = (h^g)^2_{33} = 0$, and $(h^g)^i_j = 0$ for all i, j. It now follows from the De Rham Decomposition Theorem that, on each simply-connected open set U of M, $\bar{g}$ is a product metric for some product structure on U. Note that the product structure may differ from the original one induced by the inclusion $i\colon U \to N \times S^1$ by a diffeomorphism.

We now remove the restriction that $\bar{g}_{33}$ be a function on N only.

3.5 PROPOSITION. *Let N be a surface, S^1 the unit circle, $\pi\colon N \times S^1 \to N$ the natural projection. Let t be a unit-length parameter on S^1 (i.e., $t = 0$ and $t = 1$ are identified). Then, for any Riemannian metric g on $N \times S^1$, there exists a diffeomorphism $\phi\colon N \times S^1 \to N \times S^1$ such that $\phi^*g(d/dt, d/dt) = \pi^*\lambda$, for some positive function λ on N.*

PROOF. Let (x_1, x_2, t) be a local product coordinate chart on $N \times S^1$, and let $\mu^{-2} = g(d/dt, d/dt)$. Define ϕ by

$$\phi_1(x_1, x_2, t) = x_1, \quad \phi_2(x_1, x_2, t) = x_2,$$

$$\phi_3(x_1, x_2, t) = K(x_1, x_2)\int_0^t \mu(x_1, x_2, s)ds,$$

where $K^{-1} = \int_0^1 \mu(x_1, x_2, t)\, dt$. It is clear that ϕ is a diffeomorphism, and it is easy to calculate that $\phi^*g_{33} = K^2$. But by construction, $K = \pi^*\lambda$ for some positive function λ on N. For compact, oriented 3-manifolds, Theorem 1.1 now follows from Propositions 3.4 and 3.5. If a product 3-manifold M is compact, but not oriented, we may apply our results to the orientation covering $\bar{M}$; the local diffeomorphism found there will project to a local diffeomorphism of M.

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