

BRICK DECOMPOSITIONS AND Q -MANIFOLDS

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ABSTRACT. A *brick decomposition* (respectively, *generalized brick decomposition*) of a metric space Y is a locally finite, star-finite closed cover $\{Y_\alpha\}$ such that each nonempty intersection $Y_{\alpha_1} \cap \cdots \cap Y_{\alpha_n}$, $n > 1$, is a compact AR (respectively, locally compact AR). Let K be the nerve of the decomposition $\{Y_\alpha\}$, let Q be the Hilbert cube, and $Q_0 = Q \setminus \text{point} \approx Q \times [0, 1)$. Then $Y \times Q \approx |K| \times Q$ (respectively, $Y \times Q_0 \approx |K| \times Q_0$).

1. Introduction. Borsuk [1] showed that if a finite-dimensional compactum admits a brick decomposition (a finite cover by compact AR's whose nonempty intersections are also AR's), then it has the homotopy type of the nerve of the decomposition. Holsztynski [6] later removed the finite-dimensional hypothesis. In this paper we use recent techniques and results of infinite-dimensional topology to show that a compactum Y has the *simple homotopy type* of the nerve of its brick decomposition $\{Y_i\}$, in the sense that the Q -manifold $Y \times Q$ is homeomorphic to the Q -manifold nerve $\{Y_i\} \times Q$ (Q is the Hilbert cube). This holds more generally for locally compact spaces and their locally finite decompositions.

A previous result in this direction appeared in [3], where it was shown that if X and Y admit order-isomorphic Q -factor decompositions satisfying appropriate Z -set conditions, then $X \times Q$ and $Y \times Q$ are homeomorphic. This theorem was initially used in the proof of a CE-mapping theorem for the PL category, with subsequent applications being made in the theory of hyperspaces ([5], [7]). The decomposition theorem of the present paper is an improvement on this earlier result, in that the Z -set hypotheses are dropped, and the conclusion is stated directly in terms of the nerve of a decomposition. At the same time, we obtain corresponding results for generalized brick decompositions by locally compact AR's. Some of these results (specifically, Corollary 2 below) have been motivated by decompositions of certain growth hyperspaces [4].

2. Statement of results. All spaces considered are locally compact separable metric. Let $Q_0 = Q \times [0, 1)$. The one-point compactification $Q_0 \cup \infty$ is the cone over Q , which is homeomorphic to Q , and thus Q_0 is homeomorphic to $Q \setminus \text{point}$.

DEFINITION. A Q -*decomposition* (respectively, Q_0 -*decomposition*) of a space

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Y is a locally finite, star-finite closed cover $\{Y_\alpha\}$ such that each nonempty intersection $Y_{\alpha_1} \cap \cdots \cap Y_{\alpha_n}$, $n \geq 1$, is a copy of Q (respectively, Q_0).

THEOREM. *Let $\{Y_\alpha\}$ be a Q -decomposition of Y , and let K be the nerve of $\{Y_\alpha\}$. Then $Y \times Q \approx |K| \times Q$. Similarly, if $\{Y_\alpha\}$ is a Q_0 -decomposition of Y , with nerve K , then $Y \times Q \approx |K| \times Q_0$.*

DEFINITION. A *brick decomposition* (respectively, *generalized brick decomposition*) of a space Y is a locally finite, star-finite closed cover $\{Y_\alpha\}$ such that each nonempty intersection $Y_{\alpha_1} \cap \cdots \cap Y_{\alpha_n}$, $n \geq 1$, is a compact AR (respectively, locally compact AR).

COROLLARY 1. *Let $\{Y_\alpha\}$ be a brick decomposition of Y , with nerve K . Then $Y \times Q \approx |K| \times Q$. Similarly, if $\{Y_\alpha\}$ is a generalized brick decomposition of Y , with nerve K , then $Y \times Q_0 \approx |K| \times Q_0$.*

DEFINITION. A *pointed Q -decomposition* of a pointed compactum (Y, p) is a finite cover $\{Y_i\}$ such that each Y_i contains p and each nontrivial intersection $Y_{i_1} \cap \cdots \cap Y_{i_n}$, $n \geq 1$, is a copy of Q . The *relative nerve* K of the pointed decomposition is the nerve of the cover $\{Y_i \setminus p\}$ of $Y \setminus p$.

COROLLARY 2. *Let $\{Y_i\}$ be a pointed Q -decomposition of (Y, p) , with relative nerve K , and suppose that $Y \setminus p$ is a Q -manifold. Then $Y \approx \text{cone}(|K| \times Q)$.*

3. Lemmas and proofs. If K is the nerve of a Q -decomposition $\{Y_\alpha\}$, we may consider that the decomposition elements are indexed by the vertices of K . Extending this notation, for each simplex σ of K we let Y_σ denote the corresponding nonempty intersection of decomposition elements. We say that $\{Y_\alpha\}$ is a *strong Q -decomposition* if for each pair of simplexes $\sigma, \tau \in K$ with $\sigma < \tau$ (σ a proper face of τ), the intersection Y_τ is a Z -set in Y_σ . A *strong Q_0 -decomposition* is defined similarly. The proof of the Theorem is accomplished by the following lemmas.

LEMMA 1. *Let $\{Y_\alpha\}$ be a Q -decomposition (respectively, Q_0 -decomposition) of Y . Then $Y \times Q$ admits a strong Q -decomposition (respectively, strong Q_0 -decomposition) with nerve isomorphic to the nerve of $\{Y_\alpha\}$.*

LEMMA 2. *Let $\{Y_\alpha\}$ be a strong Q -decomposition (respectively, strong Q_0 -decomposition) of Y , with nerve K . Then Y is homeomorphic to $|K| \times Q$ (respectively, $|K| \times Q_0$).*

In the proof of Lemma 1 we use Chapman's CE-mapping theorem [2], and West's sum theorem for Q -factors [8]. A map $f: X \rightarrow Y$ is a CE-map if it is a proper surjection and each point-inverse $f^{-1}(y)$ has trivial shape. The CE-mapping theorem states that if $f: X \rightarrow Y$ is a CE-map, and $X \times Q$ and $Y \times Q$ are Q -manifolds, then $X \times Q$ is homeomorphic to $Y \times Q$ and $f \times \text{id}: X \times Q \rightarrow Y \times Q$ is a near-homeomorphism. The sum theorem for Q -factors states that if $X = X_1 \cup X_2$ is a union of Q -factors whose inter-

section $X_1 \cap X_2$ is also a Q -factor, then X is a Q -factor.

In the proof of Lemma 2 we use Anderson's Z -set homogeneity theorem for Q [2], and its analogue for Q_0 . The homogeneity theorem states that a homeomorphism between Z -sets in Q can be extended to a homeomorphism of Q onto itself. The Q_0 -version is obtained by considering the one-point compactification $Q_0 \cup \infty \approx Q$, and observing that for A a Z -set in Q_0 , $A \cup \infty$ is a Z -set in $Q_0 \cup \infty$.

PROOF OF LEMMA 1. Consider $Q = \prod_1^\infty I_i$, with each $I_i = [0, 1]$. We construct a space \tilde{Y} such that $Y \times (0, \dots) \subset \tilde{Y} \subset Y \times Q$, the projection $\pi: \tilde{Y} \rightarrow Y \times (0, \dots)$ is a CE-map, and \tilde{Y} admits a strong Q -decomposition (respectively, strong Q_0 -decomposition) $\{\tilde{Y}_\alpha\}$ indexed by the vertices of K , such that $Y_\alpha \times (0, \dots) \subset \tilde{Y}_\alpha \subset Y_\alpha \times Q$ for each α . Thus $K = \text{nerve}\{\tilde{Y}_\alpha\}$.

Let $\sigma \rightarrow I_\sigma$ be a 1-1 correspondence between the simplexes of K and a subcollection of the interval factors $\{I_i\}$ of Q . For each simplex σ of K let $\tilde{Y}_\sigma = Y_\sigma \times \prod\{I_\tau | \tau \geq \sigma\} \times (0, \dots) \subset Y_\sigma \times Q$, and let $\tilde{Y} = \cup\{\tilde{Y}_\sigma | \sigma \in K\}$. Each \tilde{Y}_σ is homeomorphic to Q (respectively, Q_0), and \tilde{Y}_τ is a Z -set in \tilde{Y}_σ if $\sigma < \tau$. Since $\cap\{\tilde{Y}_\alpha | \alpha \text{ is a vertex of } \sigma\} = \tilde{Y}_\sigma$, $\{\tilde{Y}_\alpha | \alpha \text{ is a vertex of } K\}$ is a strong Q -decomposition (respectively, strong Q_0 -decomposition) of \tilde{Y} with nerve K .

An easy induction using the sum theorem for Q -factors shows that if $\{Y_\alpha\}$ is a Q -decomposition of Y , then for each $y \in Y$, $\cup\{Y_\alpha | y \in Y_\alpha\}$ is a Q -factor. Similarly, if $\{Y_\alpha\}$ is a Q_0 -decomposition of Y , then for each $y \in Y$, $\cup\{Y_\alpha \cup \infty | y \in Y_\alpha\}$ (where " $\cup \infty$ " indicates the one-point compactification) is a Q -factor, hence $\cup\{Y_\alpha \times Q | y \in Y_\alpha\}$ is a Q -manifold. Therefore, both $Y \times Q$ and $\tilde{Y} \times Q$ are Q -manifolds. Since $\pi: \tilde{Y} \rightarrow Y \times (0, \dots)$ is clearly a CE-map, there is a homeomorphism $\tilde{Y} \times Q \approx Y \times Q$ by which the decomposition $\{\tilde{Y}_\alpha \times Q\}$ is transported to the desired decomposition of $Y \times Q$.

PROOF OF LEMMA 2. A homeomorphism $h: Y \rightarrow |K| \times Q$ (respectively, $h: Y \rightarrow |K| \times Q_0$) is built up inductively, using the dual cell structure of K . Let $\text{sd } K$ be the standard barycentric subdivision. For each $\sigma \in K$ we consider the dual cell $D(\sigma) = \cap\{\text{St}(\alpha; \text{sd } K) | \alpha \text{ is a vertex of } \sigma\}$. Then $|K| = \cup\{D(\sigma) | \sigma \in K\} = \cup\{D(\alpha) | \alpha \text{ is a vertex of } K\}$, each $D(\sigma)$ is a convex cell, and $D(\tau)$ is a Z -set in $D(\sigma)$ whenever $\sigma < \tau$. Thus $\{D(\alpha) \times Q\}$ is a strong Q -decomposition of $|K| \times Q$, and $\{D(\alpha) \times Q_0\}$ is a strong Q_0 -decomposition of $|K| \times Q_0$.

We consider first the case of a strong Q -decomposition $\{Y_\alpha\}$. For each maximal simplex $\tau \in K$, choose an arbitrary homeomorphism $h_\tau: Y_\tau \rightarrow D(\tau) \times Q$. Using the Z -set homogeneity theorem, we inductively obtain homeomorphisms $h_\sigma: Y_\sigma \rightarrow D(\sigma) \times Q$, for each $\sigma \in K$, such that $h_{\sigma|Y_\tau} = h_\tau$ whenever $\sigma < \tau$. The desired homeomorphism h is obtained by setting $h(y) = h_\alpha(y)$ for $y \in Y_\alpha$. The construction in the case of a strong Q_0 -decomposition is identical, using the Z -set homogeneity theorem for Q_0 .

REMARK. Lemma 2 (in the case of a Q -decomposition) is in fact an application of the decomposition theorem of [3], since the strong Q -decomposition $\{Y_\alpha\}$ and $\{D(\alpha) \times Q\}$ are order-isomorphic in the sense of [3]. Note that the homeomorphism $h: Y \rightarrow |K| \times Q$ constructed above satisfies $h(Y_\alpha) = D(\alpha) \times Q = \text{St}(\alpha; \text{sd } K) \times Q$ for each vertex α of K . Since the near-homeomorphism $\pi \times \text{id}: \tilde{Y} \times Q \rightarrow Y \times Q$ constructed in the proof of Lemma 1 takes each $\tilde{Y}_\alpha \times Q$ onto $Y_\alpha \times Q$, it is easily seen that for $\{Y_\alpha\}$ a Q -decomposition with nerve K , there exists a homeomorphism $g: Y \times Q \rightarrow |K| \times Q$ such that $g(Y_\alpha \times Q) \subset \text{St}(\text{St}(\alpha; K); \text{sd } K)$ for each α . Similarly for a Q_0 -decomposition.

PROOF OF COROLLARY 1. Using Edwards' product theorem [2] (every locally compact metric ANR is a Q -manifold factor), and characterization theorems of Chapman [2] (every compact contractible Q -manifold is homeomorphic to Q , and every contractible Q -manifold which is $[0, 1)$ -stable is homeomorphic to Q_0), we see that $\{Y_\alpha \times Q\}$ is a Q -decomposition of $Y \times Q$ (respectively, $\{Y_\alpha \times Q_0\}$ is a Q_0 -decomposition of $Y \times Q_0$). Thus $Y \times Q \approx (Y \times Q) \times Q \approx |K| \times Q$ (respectively, $Y \times Q_0 \approx (Y \times Q_0) \times Q \approx |K| \times Q_0$).

An alternative proof for the compact case of Corollary 1, which is independent of the Theorem (and therefore gives another proof of the Theorem in the case of Q -decompositions), is also based on Edwards' product theorem and Chapman's CE-mapping theorem, but uses an intermediary space. Let $W = \cup \{Y_\sigma \times |\sigma| \mid \sigma \in K\} \subset Y \times |K|$, and let $f: W \rightarrow Y$ and $g: W \rightarrow |K|$ be the projection maps. One verifies that W is an ANR and that f and g are CE-maps, and concludes that $f \times \text{id}: W \times Q \rightarrow Y \times Q$ and $g \times \text{id}: W \times Q \rightarrow |K| \times Q$ are near-homeomorphisms.

PROOF OF COROLLARY 2. This follows from stability for the Q -manifold $Y \setminus p$, and the fact that $\{Y_\alpha \setminus p\}$ is a Q_0 -decomposition of $Y \setminus p$ with nerve K . Thus $Y \setminus p \approx (Y \setminus p) \times Q \approx |K| \times Q_0$. Forming the one-point compactification yields $Y \approx (Y \setminus p) \cup \infty \approx (|K| \times Q \times [0, 1)) \cup \infty = \text{cone}(|K| \times Q)$.

Some interesting applications of Corollary 2 have been made to growth hyperspaces [4].

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