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BRICK DECOMPOSITIONS AND Q-MANIFOLDS

D. W. CURTIS AND G. KOZLOWSKI

ABSTRACT. A *brick decomposition* (respectively, *generalized brick decomposition*) of a metric space Y is a locally finite, star-finite closed cover $\{Y_\alpha\}$ such that each nonempty intersection $Y_{\alpha_1} \cap \cdots \cap Y_{\alpha_n}$, $n > 1$, is a compact AR (respectively, locally compact AR). Let K be the nerve of the decomposition $\{Y_\alpha\}$, let Q be the Hilbert cube, and $Q_0 = Q \setminus \text{point} \approx Q \times [0, 1)$. Then $Y \times Q \approx |K| \times Q$ (respectively, $Y \times Q_0 \approx |K| \times Q_0$).

1. Introduction. Borsuk [1] showed that if a finite-dimensional compactum admits a brick decomposition (a finite cover by compact AR's whose nonempty intersections are also AR's), then it has the homotopy type of the nerve of the decomposition. Holsztynski [6] later removed the finite-dimensional hypothesis. In this paper we use recent techniques and results of infinite-dimensional topology to show that a compactum Y has the *simple homotopy type* of the nerve of its brick decomposition $\{Y_i\}$, in the sense that the Q-manifold $Y \times Q$ is homeomorphic to the Q-manifold nerve $\{Y_i\} \times Q$ (Q is the Hilbert cube). This holds more generally for locally compact spaces and their locally finite decompositions.

A previous result in this direction appeared in [3], where it was shown that if X and Y admit order-isomorphic Q-factor decompositions satisfying appropriate Z-set conditions, then $X \times Q$ and $Y \times Q$ are homeomorphic. This theorem was initially used in the proof of a CE-mapping theorem for the PL category, with subsequent applications being made in the theory of hyperspaces ([5], [7]). The decomposition theorem of the present paper is an improvement on this earlier result, in that the Z-set hypotheses are dropped, and the conclusion is stated directly in terms of the nerve of a decomposition. At the same time, we obtain corresponding results for generalized brick decompositions by locally compact AR's. Some of these results (specifically, Corollary 2 below) have been motivated by decompositions of certain growth hyperspaces [4].

2. Statement of results. All spaces considered are locally compact separable metric. Let $Q_0 = Q \times [0, 1)$. The one-point compactification $Q_0 \cup \infty$ is the cone over Q, which is homeomorphic to Q, and thus Q_0 is homeomorphic to $Q \setminus$ point.

DEFINITION. A *Q-decomposition* (respectively, *Q_0-decomposition*) of a space

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Y is a locally finite, star-finite closed cover $\{Y_\alpha\}$ such that each nonempty intersection $Y_{\alpha_1} \cap \cdots \cap Y_{\alpha_n}$, $n \geq 1$, is a copy of Q (respectively, Q_0).

THEOREM. *Let $\{Y_\alpha\}$ be a Q-decomposition of Y, and let K be the nerve of $\{Y_\alpha\}$. Then $Y \times Q \approx |K| \times Q$. Similarly, if $\{Y_\alpha\}$ is a Q_0-decomposition of Y, with nerve K, then $Y \times Q \approx |K| \times Q_0$.*

DEFINITION. A *brick decomposition* (respectively, *generalized brick decomposition*) of a space Y is a locally finite, star-finite closed cover $\{Y_\alpha\}$ such that each nonempty intersection $Y_{\alpha_1} \cap \cdots \cap Y_{\alpha_n}$, $n \geq 1$, is a compact AR (respectively, locally compact AR).

COROLLARY 1. *Let $\{Y_\alpha\}$ be a brick decomposition of Y, with nerve K. Then $Y \times Q \approx |K| \times Q$. Similarly, if $\{Y_\alpha\}$ is a generalized brick decomposition of Y, with nerve K, then $Y \times Q_0 \approx |K| \times Q_0$.*

DEFINITION. A *pointed Q-decomposition* of a pointed compactum (Y, p) is a finite cover $\{Y_i\}$ such that each Y_i contains p and each nontrivial intersection $Y_{i_1} \cap \cdots \cap Y_{i_n}$, $n \geq 1$, is a copy of Q. The *relative nerve* K of the pointed decomposition is the nerve of the cover $\{Y_i \setminus p\}$ of $Y \setminus p$.

COROLLARY 2. *Let $\{Y_i\}$ be a pointed Q-decomposition of (Y, p), with relative nerve K, and suppose that $Y \setminus p$ is a Q-manifold. Then $Y \approx$ cone$(|K| \times Q)$.*

3. Lemmas and proofs. If K is the nerve of a Q-decomposition $\{Y_\alpha\}$, we may consider that the decomposition elements are indexed by the vertices of K. Extending this notation, for each simplex σ of K we let Y_σ denote the corresponding nonempty intersection of decomposition elements. We say that $\{Y_\alpha\}$ is a *strong* Q-decomposition if for each pair of simplexes $\sigma, \tau \in K$ with $\sigma < \tau$ (σ a proper face of τ), the intersection Y_τ is a Z-set in Y_σ. A *strong* Q_0-decomposition is defined similarly. The proof of the Theorem is accomplished by the following lemmas.

LEMMA 1. *Let $\{Y_\alpha\}$ be a Q-decomposition* (*respectively*, *Q_0-decomposition*) *of Y. Then $Y \times Q$ admits a strong Q-decomposition* (*respectively*, *strong Q_0-decomposition*) *with nerve isomorphic to the nerve of $\{Y_\alpha\}$.*

LEMMA 2. *Let $\{Y_\alpha\}$ be a strong Q-decomposition* (*respectively*, *strong Q_0-decomposition*) *of Y, with nerve K. Then Y is homeomorphic to $|K| \times Q$* (*respectively*, *$|K| \times Q_0$*).

In the proof of Lemma 1 we use Chapman's CE-mapping theorem [**2**], and West's sum theorem for Q-factors [**8**]. A map $f\colon X \to Y$ is a CE-map if it is a proper surjection and each point-inverse $f^{-1}(y)$ has trivial shape. The CE-mapping theorem states that if $f\colon X \to Y$ is a CE-map, and $X \times Q$ and $Y \times Q$ are Q-manifolds, then $X \times Q$ is homeomorphic to $Y \times Q$ and $f \times \mathrm{id}\colon X \times Q \to Y \times Q$ is a near-homeomorphism. The sum theorem for Q-factors states that if $X = X_1 \cup X_2$ is a union of Q-factors whose intersection $X_1 \cap X_2$ is also a Q-factor, then X is a Q-factor.

In the proof of Lemma 2 we use Anderson's Z-set homogeneity theorem for Q [2], and its analogue for Q_0. The homogeneity theorem states that a homeomorphism between Z-sets in Q can be extended to a homeomorphism of Q onto itself. The Q_0-version is obtained by considering the one-point compactification $Q_0 \cup \infty \approx Q$, and observing that for A a Z-set in Q_0, $A \cup \infty$ is a Z-set in $Q_0 \cup \infty$.

PROOF OF LEMMA 1. Consider $Q = \prod_1^\infty I_i$, with each $I_i = [0, 1]$. We construct a space $\tilde{Y}$ such that $Y \times (0, \dots) \subset \tilde{Y} \subset Y \times Q$, the projection π: $\tilde{Y} \to Y \times (0, \dots)$ is a CE-map, and $\tilde{Y}$ admits a strong Q-decomposition (respectively, strong Q_0-decomposition) $\{\tilde{Y}_\alpha\}$ indexed by the vertices of K, such that $Y_\alpha \times (0, \dots) \subset \tilde{Y}_\alpha \subset Y_\alpha \times Q$ for each α. Thus $K = \text{nerve}\{\tilde{Y}_\alpha\}$.

Let $\sigma \to I_\sigma$ be a 1-1 correspondence between the simplexes of K and a subcollection of the interval factors $\{I_i\}$ of Q. For each simplex σ of K let $\tilde{Y}_\sigma = Y_\sigma \times \prod\{I_\tau | \tau \geqslant \sigma\} \times (0, \dots) \subset Y_\sigma \times Q$, and let $\tilde{Y} = \bigcup\{\tilde{Y}_\sigma | \sigma \in K\}$. Each $\tilde{Y}_\sigma$ is homeomorphic to Q (respectively, Q_0), and $\tilde{Y}_\tau$ is a Z-set in $\tilde{Y}_\sigma$ if $\sigma < \tau$. Since $\bigcap\{\tilde{Y}_\alpha | \alpha \text{ is a vertex of } \sigma\} = \tilde{Y}_\sigma$, $\{\tilde{Y}_\alpha | \alpha \text{ is a vertex of } K\}$ is a strong Q-decomposition (respectively, strong Q_0-decomposition) of $\tilde{Y}$ with nerve K.

An easy induction using the sum theorem for Q-factors shows that if $\{Y_\alpha\}$ is a Q-decomposition of Y, then for each $y \in Y$, $\bigcup\{Y_\alpha | y \in Y_\alpha\}$ is a Q-factor. Similarly, if $\{Y_\alpha\}$ is a Q_0-decomposition of Y, then for each $y \in Y$, $\bigcup\{Y_\alpha \cup \infty | y \in Y_\alpha\}$ (where "$\cup \infty$" indicates the one-point compactification) is a Q-factor, hence $\bigcup\{Y_\alpha \times Q | y \in Y_\alpha\}$ is a Q-manifold. Therefore, both $Y \times Q$ and $\tilde{Y} \times Q$ are Q-manifolds. Since π: $\tilde{Y} \to Y \times (0, \dots)$ is clearly a CE-map, there is a homeomorphism $\tilde{Y} \times Q \approx Y \times Q$ by which the decomposition $\{\tilde{Y}_\alpha \times Q\}$ is transported to the desired decomposition of $Y \times Q$.

PROOF OF LEMMA 2. A homeomorphism h: $Y \to |K| \times Q$ (respectively, h: $Y \to |K| \times Q_0$) is built up inductively, using the dual cell structure of K. Let $\operatorname{sd} K$ be the standard barycentric subdivision. For each $\sigma \in K$ we consider the dual cell $D(\sigma) = \bigcap\{\operatorname{St}(\alpha; \operatorname{sd} K) | \alpha \text{ is a vertex of } \sigma\}$. Then $|K| = \bigcup\{D(\sigma) | \sigma \in K\} = \bigcup\{D(\alpha) | \alpha \text{ is a vertex of } K\}$, each $D(\sigma)$ is a convex cell, and $D(\tau)$ is a Z-set in $D(\sigma)$ whenever $\sigma < \tau$. Thus $\{D(\alpha) \times Q\}$ is a strong Q-decomposition of $|K| \times Q$, and $\{D(\alpha) \times Q_0\}$ is a strong Q_0-decomposition of $|K| \times Q_0$.

We consider first the case of a strong Q-decomposition $\{Y_\alpha\}$. For each maximal simplex $\tau \in K$, choose an arbitrary homeomorphism h_τ: $Y_\tau \to D(\tau) \times Q$. Using the Z-set homogeneity theorem, we inductively obtain homeomorphisms h_σ: $Y_\sigma \to D(\sigma) \times Q$, for each $\sigma \in K$, such that $h_{\sigma | Y_\tau} = h_\tau$ whenever $\sigma < \tau$. The desired homeomorphism h is obtained by setting $h(y) = h_\alpha(y)$ for $y \in Y_\alpha$. The construction in the case of a strong Q_0-decomposition is identical, using the Z-set homogeneity theorem for Q_0.

REMARK. Lemma 2 (in the case of a Q-decomposition) is in fact an application of the decomposition theorem of **[3]**, since the strong Q-decomposition $\{Y_\alpha\}$ and $\{D(\alpha) \times Q\}$ are order-isomorphic in the sense of [3]. Note that the homeomorphism h: $Y \to |K| \times Q$ constructed above satisfies $h(Y_\alpha) = D(\alpha) \times Q = \mathrm{St}(\alpha;\ \mathrm{sd}\ K) \times Q$ for each vertex α of K. Since the near-homeomorphism $\pi \times \mathrm{id}$: $\tilde{Y} \times Q \to Y \times Q$ constructed in the proof of Lemma 1 takes each $\tilde{Y}_\alpha \times Q$ onto $Y_\alpha \times Q$, it is easily seen that for $\{Y_\alpha\}$ a Q-decomposition with nerve K, there exists a homeomorphism g: $Y \times Q \to |K| \times Q$ such that $g(Y_\alpha \times Q) \subset \mathrm{St}(\mathrm{St}(\alpha;\ K);\ \mathrm{sd}\ K)$ for each α. Similarly for a Q_0-decomposition.

PROOF OF COROLLARY 1. Using Edwards' product theorem **[2]** (every locally compact metric ANR is a Q-manifold factor), and characterization theorems of Chapman **[2]** (every compact contractible Q-manifold is homeomorphic to Q, and every contractible Q-manifold which is $[0, 1)$-stable is homeomorphic to Q_0), we see that $\{Y_\alpha \times Q\}$ is a Q-decomposition of $Y \times Q$ (respectively, $\{Y_\alpha \times Q_0\}$ is a Q_0-decomposition of $Y \times Q_0$). Thus $Y \times Q \approx (Y \times Q) \times Q \approx |K| \times Q$ (respectively, $Y \times Q_0 \approx (Y \times Q_0) \times Q \approx |K| \times Q_0$).

An alternative proof for the compact case of Corollary 1, which is independent of the Theorem (and therefore gives another proof of the Theorem in the case of Q-decompositions), is also based on Edwards' product theorem and Chapman's CE-mapping theorem, but uses an intermediary space. Let $W = \bigcup\{Y_\sigma \times |\sigma|\ |\sigma \in K\} \subset Y \times |K|$, and let f: $W \to Y$ and g: $W \to |K|$ be the projection maps. One verifies that W is an ANR and that f and g are CE-maps, and concludes that $f \times \mathrm{id}$: $W \times Q \to Y \times Q$ and $g \times \mathrm{id}$: $W \times Q \to |K| \times Q$ are near-homeomorphisms.

PROOF OF COROLLARY 2. This follows from stability for the Q-manifold $Y \setminus p$, and the fact that $\{Y_\alpha \setminus p\}$ is a Q_0-decomposition of $Y \setminus p$ with nerve K. Thus $Y \setminus p \approx (Y \setminus p) \times Q \approx |K| \times Q_0$. Forming the one-point compactification yields $Y \approx (Y \setminus p) \cup \infty \approx (|K| \times Q \times [0, 1)) \cup \infty = \mathrm{cone}(|K| \times Q)$.

Some interesting applications of Corollary 2 have been made to growth hyperspaces **[4]**.

REFERENCES

1. K. Borsuk, *On the imbedding of systems of compacta in simplicial complexes*, Fund. Math. **35** (1948), 217–234.

2. T. A. Chapman, *Lectures on Hilbert cube manifolds*, CBMS Regional Conf. Ser. in Math., no. 28, Amer. Math. Soc., Providence, R. I., 1976.

3. D. W. Curtis, *Simplicial maps which stabilize to near-homeomorphisms*, Compositio Math. **25** (1972), 117–122.

4. ______, *Growth hyperspaces of Peano continua*, Trans. Amer. Math. Soc. **238** (1978), 271–283.

5. D. W. Curtis and R. M. Schori, *Hyperspaces which characterize simple homotopy type*, General Topology and Appl. **6** (1976), 153–165.

6. W. Holsztynski, *On spaces with regular decompositions*, Bull. Acad. Polon. Sci. **12** (1964), 607–611.

7. R. M. Schori and J. E. West, *The hyperspace of the closed unit interval is a Hilbert cube*, Trans. Amer. Math. Soc. **213** (1975), 217–235.

8. J. E. West, *Sums of Hilbert cube factors*, Pacific J. Math. **54** (1974), 293–303.

DEPARTMENT OF MATHEMATICS, LOUISIANA STATE UNIVERSITY, BATON ROUGE, LOUISIANA 70803

DEPARTMENT OF MATHEMATICS, AUBURN UNIVERSITY, AUBURN, ALABAMA 36830