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ON THE PROXIMINALITY OF COMPACT OPERATORS WITH RANGE IN $C(S)$

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ABSTRACT. It is proved that the space of compact operators on X into $C(S)$ is proximinal in the corresponding space of bounded operators, if X is a Hilbert space, a space l_p, $1 < p < +\infty$, or the space c_0.

Let X be a Banach space, E a closed subspace of X, $x \in X$. We denote by $P_E(x)$ the set of all elements of best approximation of x in E, i.e., $P_E(x) = \{e \in E;\ \|x - e\| = \inf_{y \in E}\|x - y\|\}$. E is said to be proximinal, if $P_E(x) \neq \emptyset$ for every $x \in X$.

Recently, many authors have investigated the problem of best approximation of bounded linear operators $B(X, Y)$ on a Banach space X into a Banach space Y by elements of the corresponding space of compact operators $K(X, Y)$ (see, e.g., [2]–[5]). Although, in some special cases, a lot of information is known, the following problem still remains open, even for a number of standard Banach spaces: For which Banach spaces X and Y is $K(X, Y)$ proximinal in $B(X, Y)$? The purpose of this paper is to give an affirmative answer to this question in the case that X is a Hilbert space, a space l_p, $1 < p < +\infty$, or the space c_0, and Y is $C(S)$, the space of all continuous scalar functions on a compact Hausdorff space S.

To obtain one general proof valid for all mentioned cases we make the following definitions.

DEFINITION 1. A Banach space X is said to have the property (P_1) if there is a continuous strictly increasing function Ψ: $\mathbf{R}^+ \to \mathbf{R}^+$ with $\Psi(0) = 0$ and $\Psi(1) = 1$ such that the following is true: For every bounded net $\{x_\alpha\}$ in the dual X^* of X w^*-converging to 0, every $y \in X^*$ and each $\varepsilon > 0$ there exists an α_0 such that

$$|\Psi(\|x_\alpha + y\|) - \Psi(\|x_\alpha\|) - \Psi(\|y\|)| < \varepsilon \tag{1}$$

for every $\alpha \geqslant \alpha_0$.

DEFINITION 2. A Banach space X is said to have the property (P_2) if for every bounded net $\{x_\alpha\}$ in X^* with w^*-$\lim x_\alpha = 0$ and $\lim\|x_\alpha\| = 1$, every $y \in X^*$ with $\|y\| \leqslant 1$ and every $\varepsilon > 0$ there is a cofinal subnet $\{x_\beta\}$ of $\{x_\alpha\}$ such that for every β

$$\|x_\beta - y\| \geqslant 1 - \varepsilon.$$

The following is obvious.

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PROPOSITION 1. *Every Banach space with the property* (P_1) *has the property* (P_2).

Now we give some examples.

PROPOSITION 2. *Every Hilbert space H has the property* (P_1).

PROOF. Putting $\Psi(t) = t^2$, $t \in \mathbf{R}^+$, for any net $\{x_\alpha\} \subset H$ w^*-converging to 0 and any $y \in H$, the claim follows from the relation

$$\Psi(\|x_\alpha + y\|) = \Psi(\|x_\alpha\|) + \Psi(\|y\|) + (x_\alpha, y) + (y, x_\alpha).$$

PROPOSITION 3. *Every space l_p, $1 < p < +\infty$, has the property* (P_1). *The space c_0 has the property* (P_1).

PROOF. For $1 \leqslant q < +\infty$ put $\Psi(t) = t^q$. Let $\varepsilon > 0$, $x_\alpha = \{\xi_i^\alpha\} \in l_q$ with w^*-$\lim x_\alpha = 0$, and $y = \{\eta_i\} \in l_q$ be given. Put $M = \sup\|x_\alpha\|$. Since Ψ is uniformly continuous in $[0, M + 1]$, there is an $n_0 \in \mathbf{N}$ such that

$$\left| \sum_{i=n_0+1}^{\infty} |\xi_i^\alpha + \eta_i|^q - \sum_{i=n_0+1}^{\infty} |\xi_i^\alpha|^q \right| < \varepsilon/4$$

for every α, and such that

$$\sum_{i=n_0+1}^{\infty} |\eta_i|^q < \varepsilon/4.$$

Find an α_0 such that

$$\left|\, |\xi_i^\alpha + \eta_i|^q - |\eta_i|^q \,\right| < \varepsilon/4n_0$$

for $\alpha \geqslant \alpha_0$, $1 \leqslant i \leqslant n_0$, and

$$\sum_{i=1}^{n_0} |\xi_i^\alpha|^q \leqslant \varepsilon/4$$

for $\alpha \geqslant \alpha_0$. It follows that

$$|\Psi(\|x_\alpha + y\|) - \Psi(\|x_\alpha\|) - \Psi(\|y\|)|$$
$$= \left| \sum_{i=1}^{\infty} |\xi_i^\alpha + \eta_i|^q - \sum_{i=1}^{\infty} |\xi_i^\alpha|^q - \sum_{i=1}^{\infty} |\eta_i|^q \right| < \varepsilon.$$

This completes the proof.

Let X be a Banach space. By the representation theorem VI.7.1 of **[1]**, there is a one-to-one correspondence between $B(X, C(S))$ and w^*-continuous maps $u: S \to X^*$ such that for $L \in B(X, C(X))$

$$Lx(t) = u(t)x, \qquad x \in X, t \in S,$$

and

$$\|L\| = \sup_{t \in S} \|u(t)\|.$$

The operator L is compact iff u is continuous with the norm topology in X^*.

In the next theorem a lower bound for the distance

$$\operatorname{dist}\big(L, K(X, C(S))\big) = \inf_{T \in K(X,C(S))} \|L - T\|$$

of an arbitrary $L \in B(X, C(S))$ to $K(X, C(S))$ is given.

THEOREM 1. *Let $L \in B(X, C(S))$, where X has the property* (P_2). *Let u: $S \to X^*$ be the corresponding w^*-continuous map. Then we have*

$$\operatorname{dist}\big(L, K(X, C(S))\big) \geqslant \sup_{t \in S} \limsup_{s \to t} \|u(t) - u(s)\|.$$

PROOF. For $t \in S$ denote $d(t) = \limsup_{s \to t}\|u(t) - u(s)\|$. For an operator $T \in K(X, C(S))$ let v: $S \to X^*$ be the corresponding map. We have to show that for every $t \in S$

$$\|L - T\| = \sup_{s \in S} \|u(s) - v(s)\| \geqslant d(t),$$

or, in other words, that for every $t \in S$ and every $\varepsilon > 0$ there is an $s \in S$ such that

$$\|u(s) - v(s)\| \geqslant d(t) - \varepsilon.$$

Suppose that this is not true. Then there are a $t_0 \in S$ and an $\varepsilon_0 > 0$ such that for every $s \in S$ we have

$$\|u(s) - v(s)\| < d(t_0) - \varepsilon_0. \tag{2}$$

There is a net $\{s_\alpha\} \subset S$ with $\lim s_\alpha = t_0$ and

$$\lim\|u(s_\alpha) - u(t_0)\| = d(t_0). \tag{3}$$

Since v is continuous with the norm topology in X^*, there is an α_0 such that for every $\alpha \geqslant \alpha_0$ we have

$$\|v(s_\alpha) - v(t_0)\| < \varepsilon_0/2. \tag{4}$$

Consequently, by (2) and (4),

$$\begin{aligned}\|u(s_\alpha) - v(t_0)\| &\leqslant \|u(s_\alpha) - v(s_\alpha)\| + \|v(s_\alpha) - v(t_0)\| \\ &< d(t_0) - \varepsilon_0/2\end{aligned} \tag{5}$$

for every $\alpha \geqslant \alpha_0$. Denoting for $s \in S$

$$u_1(s) = (1/d(t_0))u(s), \qquad v_1(s) = (1/d(t_0))(v(s) - u(s)),$$

we obtain $w^*\text{-}\lim(u_1(s_\alpha) - u_1(t_0)) = 0$ and, by (3), $\lim\|u_1(s_\alpha) - u_1(t_0)\| = 1$. Further, by (5), we have for $\alpha \geqslant \alpha_0$

$$\begin{aligned}&\|u_1(s_\alpha) - u_1(t_0) - v_1(t_0)\| \\ &\quad = (1/d(t_0))\|u(s_\alpha) - u(t_0) - v(t_0) + u(t_0)\| \\ &\quad = (1/d(t_0))\|u(s_\alpha) - v(t_0)\| < 1 - \varepsilon_0/\big(2d(t_0)\big).\end{aligned}$$

Hence X cannot have the property (P_2) which contradicts the assumption.

Now we formulate our main result.

THEOREM 2. *If a Banach space X has the property* (P_1), *then $K(X, C(S))$ is proximinal in $B(X, C(S))$.*

PROOF. Let L be an arbitrary bounded operator in $B(X, C(S))$, $u: S \to X^*$ the corresponding w^*-continuous map. Denote $d(t) = \limsup_{s \to t} \|u(t) - u(s)\|$, $d = \sup_{t \in S} d(t)$. By Theorem 1 it is sufficient to show that there exists a function $v: S \to X^*$ which is continuous with the norm topology in X^* such that

$$\sup_{t \in S} \|v(t) - u(t)\| \leqslant d. \tag{6}$$

Since X has the property (P_1), there is a strictly increasing continuous function $\Psi: \mathbf{R}^+ \to \mathbf{R}^+$ with $\Psi(0) = 0$ and $\Psi(1) = 1$ satisfying (1). Define a set valued mapping $\Phi: S \to 2^{X^*}$ by

$$\Phi(t) = \left\{x \in X^*; \|x - u(t)\| \leqslant \Psi^{-1}\big(\Psi(d) - \Psi(d(t))\big)\right\}.$$

To accomplish the proof we need the following two lemmas.

LEMMA 1. *For every $t \in S$ and every $\varepsilon > 0$ there is a neighborhood U of t such that for every $s \in U$ we have*

$$\Psi(d(t)) \geqslant \Psi(d(s)) + \Psi(\|u(t) - u(s)\|) - \varepsilon.$$

PROOF. Assume that there is a $t \in S$ and an $\varepsilon_0 > 0$ such that every neighborhood of t contains a point s with

$$\Psi(d(t)) < \Psi(d(s)) + \Psi(\|u(t) - u(s)\|) - \varepsilon_0. \tag{7}$$

There is a neighborhood U_0 of t such that for every $s \in U_0$ we have

$$\Psi(\|u(t) - u(s)\|) \leqslant \Psi(d(t)) + \varepsilon_0/6. \tag{8}$$

Find an $s_0 \in U_0$ for which (7) holds. Further, there is an $s_1 \in U_0$ such that

$$\Psi(\|u(s_0) - u(s_1)\|) \geqslant \Psi(d(s_0)) - \varepsilon_0/6, \tag{9}$$

and, since u is w^*-continuous, such that

$$\begin{aligned}\Psi(\|u(t) - u(s_0) + u(s_0) - u(s_1)\|)& \\ \geqslant \Psi(\|u(t) - u(s_0)\|) + \Psi(\|u(s_0) - u(s_1)\|) - \varepsilon_0/6.& \quad (10)\end{aligned}$$

Consequently, by (7)–(10),

$$\begin{aligned}\Psi(d(s_0)) + \Psi(\|u(t) - u(s_0)\|) - \varepsilon_0 &> \Psi(d(t)) \\ &\geqslant \Psi(\|u(t) - u(s_1)\|) - \varepsilon_0/6 \\ &\geqslant \Psi(\|u(t) - u(s_0)\|) + \Psi(\|u(s_0) - u(s_1)\|) - 2\varepsilon_0/6 \\ &\geqslant \Psi(\|u(t) - u(s_0)\|) + \Psi(d(s_0)) - \varepsilon_0/2.\end{aligned}$$

It follows that $-\varepsilon_0/2 > 0$. A contradiction.

LEMMA 2. *For every $t \in S$ and every $\varepsilon > 0$ there is a neighborhood U of t such that for every $s \in U$ and every $x \in X^*$*

$$\|x - u(t)\| \leqslant \Psi^{-1}\big(\Psi(d) - \Psi(d(t))\big)$$

implies

$$\|x - u(s)\| \leqslant \Psi^{-1}\big(\Psi(d) - \Psi(d(s))\big) + \varepsilon.$$

PROOF. Given $\varepsilon > 0$ there is, since Ψ^{-1} is uniformly continuous in $[0, \Psi(d) + 1]$, a $\delta > 0$ such that for every $t_1, t_2 \in [0, \Psi(d) + 1]$ $t_1 \leqslant t_2 + \delta$ implies

$$\Psi^{-1}(t_1) \leqslant \Psi^{-1}(t_2) + \varepsilon. \tag{11}$$

By Lemma 1 there is a neighborhood U_1 of t such that for every $s \in U_1$ we have

$$\Psi(d(t)) \geqslant \Psi(d(s)) + \Psi(\|u(t) - u(s)\|) - \delta/2. \tag{12}$$

Further, since u is w^*-continuous, there is a neighborhood U_2 of t such that

$$\begin{aligned}\Psi(\|x &- u(t) + u(t) - u(s)\|)\\ &\leqslant \Psi(\|x - u(t)\|) + \Psi(\|u(t) - u(s)\|) + \delta/2\end{aligned} \tag{13}$$

for every $s \in U_2$. Then, by (12) and (13), for each $s \in U = U_1 \cap U_2$

$$\begin{aligned}\Psi(\|x - u(s)\|) &= \Psi(\|x - u(t) + u(t) - u(s)\|)\\ &\leqslant \Psi(\|x - u(t)\|) + \Psi(\|u(t) - u(s)\|) + \delta/2\\ &\leqslant \Psi(d) - \Psi(d(t)) + \Psi(\|u(t) - u(s)\|) + \delta/2\\ &\leqslant \Psi(d) - \Psi(d(s)) + \delta.\end{aligned}$$

Consequently, by (11),

$$\|x - u(s)\| \leqslant \Psi^{-1}\big(\Psi(d) - \Psi(d(s))\big) + \varepsilon, \qquad s \in U,$$

which completes the proof of Lemma 2.

Now, we can accomplish the proof of Theorem 2. It follows from Lemma 2 that Φ is lower semicontinuous (cf. [7, Proposition 2.1]). Hence, by E. Michael's selection Theorem 3.2″ of [7], Φ has a continuous selection v which obviously fulfils (6). This completes the proof of Theorem 2.

COROLLARY 3. *If X is a Hilbert space, a space l_p, $1 < p < +\infty$, or the space c_0, then $K(X, C(S))$ is proximinal in $B(X, C(S))$.*

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