

ON THE SIERPIŃSKI-ERDÖS
AND THE OXTOBY-ULAM THEOREMS
FOR SOME NEW SIGMA-IDEALS OF SETS^{1,2}

C. G. MENDEZ

ABSTRACT. Let $\Phi(\Psi)$ denote the family of subsets of the unit square defined to be of first category (Lebesgue measure zero) in almost every vertical line in the sense of measure (category). **THEOREM 1.** *There is a homeomorphism of the unit square onto itself mapping a given set in $\Phi(\Psi)$ onto a set of Lebesgue measure zero.* **THEOREM 2.** *There is a set belonging to both Φ and Ψ that cannot be mapped onto a set of first category by a homeomorphism of the unit square onto itself.*

Let C denote the Cantor set, regarded as the product of a sequence of 2-element groups, and let Λ denote one of the σ -ideals of subsets of C studied by Schmidt and Mycielski. **THEOREM 3.** *Assuming the continuum hypothesis, the Sierpiński-Erdős theorem holds for Λ and the class of subsets of C of Haar measure zero (or of first category).* **THEOREM 4.** *The Oxtoby-Ulam theorem holds for the image of Λ under the Cantor mapping of C onto the unit interval.*

1. On the Sierpiński-Erdős theorem. This section is inspired by the work of Schmidt [8] and Mycielski [4] on σ -ideals of sets on C (the Cantor set), or for that matter on I (the unit interval).

In correspondence with Mycielski's definition we have: Given a set $S \subset C$ and a set K of natural numbers we define a positional game $\Gamma(S, K)$ with perfect information between two players I and II. The players choose consecutive terms of a sequence $(x_0, x_1, x_2, \dots) \in C$ (where C is hereby regarded as the topological space $(\prod_{i=1}^{\infty} X_i, \prod_{i=1}^{\infty} T_i)$, $X_i = \{0, 1\}$, and T_i is the discrete topology on X_i), the choice x_i is made by player I if $i \notin K$, and by player II if $i \in K$. The player choosing x_i knows S, K , and x_0, x_1, \dots, x_{i-1} .

Player I wins if $(x_0, x_1, x_2, \dots) \in S$, and player II wins in the other case.

Let $W_{II}(K)$ be the class of sets $S \subset C$ for which player II has a winning strategy in the game $\Gamma(S, K)$. Let $M = \{K_{s_1, s_2, \dots, s_n} : s_i = 0, 1; 1 \leq i \leq n, n \in J\}$ be a system of sets of natural numbers such that $K_{s_1, s_2, \dots, s_n, s_{n+1}}$ is

Presented to the Society, April 16, 1978; received by the editors October 2, 1976 and, in revised form, December 14, 1977.

AMS (MOS) subject classifications (1970). Primary 54A05; Secondary 28A65, 90D05.

Key words and phrases. Measure zero, first category, ideals of sets, infinite games, winning strategies, homeomorphisms, Sierpiński-Erdős theorem, Oxtoby-Ulam theorem.

¹Some of the results in this article are contained in the author's Ph.D. dissertation submitted to the University of Colorado, Boulder, in December of 1974 and written under the direction of Professor S. M. Ulam.

²The author wishes to acknowledge a debt of gratitude to the Society's referee who reviewed this manuscript for her/his invaluable help and suggestions.

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contained in K_{s_1, s_2, \dots, s_n} and $K_{s_1, s_2, \dots, s_{n-1}, 0} \cap K_{s_1, s_2, \dots, s_{n-1}, 1} = \emptyset$ for $n \in J$. Let $\Lambda = \bigcap W_{II}(K_{s_1, s_2, \dots, s_n})$ where the intersection runs over all the sets K_{s_1, s_2, \dots, s_n} of the system M . We shall assume that all sets K_{s_1, s_2, \dots, s_n} of the system M are nonempty and hence infinite.

Now we would like to show the following properties for Λ .

1.1 THEOREM. (i) Λ is σ -ideal.

(ii) The union of Λ is C .

(iii) The complement of each member of Λ contains a set of power c belonging to Λ .

(iv) C may be represented as the union of two complementary Borel sets one in Λ and the other one of first category and measure zero where the measure in C is the Haar measure; or equivalently, the measure induced by the Lebesgue measure in the unit interval I and the Cantor mapping of C onto I .

(v) There is a subclass Υ of power $\leq c$ of Λ such that each member of Λ is contained in some member of the subclass Υ .

PROOF. (i) Shown in Mycielski's paper as Theorem 3.

(ii) Obvious from Theorem 4 in Mycielski's paper.

(iii) Given $S \in \Lambda$ we define the set A as follows:

$A = \{x: x = (x_0, x_1, x_2, \dots), x_j^0 = f_j^0(x_0, x_1, x_2, \dots, x_{j-1}^0)\}$ where $K_{s_1, s_2, \dots, s_n} = \{j_k^{s_1, s_2, \dots, s_n}\}$ may be thought of as a well ordered subset of the set of natural numbers and $\{f_j^0\}_{j_k^0 \in K_0}$ is a winning strategy for player II in game $\Gamma(S, K_0)$, and $x_{j_n^{s_1, s_2, \dots, s_n}}^0 = 0$ for all $n \in J$ and $\{1, s_2, \dots, s_n\}$.

We would like to show:

(a) $A \subset S^c$,

(b) $A \in \Lambda$,

(c) A is of power c .

PROOF OF (a). $x = (x_0, x_1, \dots) \in A \Rightarrow x_j^0 = f_j^0(x_0, x_1, \dots, x_{j-1}^0)$ where $\{f_j^0\}_{j_k^0 \in K_0}$ is a winning strategy for player II in game $\Gamma(S, K_0)$. (This is possible for there is such a winning strategy for $S \in \Lambda$.) Hence $x \in S^c$.

PROOF OF (b). To show that for any given $K_{s_1, s_2, \dots, s_n} \in M$, there is a winning strategy $\{g_j\}_{j \in K_{s_1, s_2, \dots, s_n}}$ for player II in game $\Gamma(A, K_{s_1, s_2, \dots, s_n})$. If

$$K_{s_1, s_2, \dots, s_n} \subset K_0 (K_{s_1, s_2, \dots, s_n} = K_{0, s_2, \dots, s_n}),$$

then $j_1^{0, s_2, \dots, s_n} = j_i^0$ for some i ; and we define g_j such that

$$\begin{aligned} x_j^0 &= x_{j_1^{0, s_2, \dots, s_n}}^0 = g_{j_1^{0, s_2, \dots, s_n}}(x_0, \dots, x_{j_1^{0, s_2, \dots, s_n}-1}) \\ &= g_{j_i^0}(x_0, \dots, x_{j_i^0-1}) \neq f_{j_i^0}(x_0, \dots, x_{j_i^0-1}) \end{aligned}$$

for that i , hence $x = (x_0, \dots, x_{j_i^0-1}, x_{j_i^0}, \dots) \notin A$.

On the other hand, if $K_{s_1, s_2, \dots, s_n} \in K_1 (K_{s_1, s_2, \dots, s_n} = K_{1, s_2, \dots, s_n})$, then define

$$x_{j_n^{1, s_2, \dots, s_n}}^0 = g_{j_n^{1, s_2, \dots, s_n}}(x_0, x_1, \dots, x_{j_n^{1, s_2, \dots, s_n}-1}) = 1 \neq 0,$$

hence $x = (x_0, x_1, \dots, x_{j_n^{1, s_2, \dots, s_n}}, \dots) \notin A$.

PROOF OF (c). Define

$$I_n = \bigcup_{1, s_2, \dots, s_n: K_{1, s_2, \dots, s_n} \in M} \{j_k^{1, s_2, \dots, s_n}: k < n \text{ and } x \in A \Rightarrow x_{j_k^{1, s_2, \dots, s_n}} = 0\}$$

and

$$J_n = \bigcup_{1, s_2, \dots, s_n: K_{1, s_2, \dots, s_n} \in M} \{j_k^{1, s_2, \dots, s_n}: k < n\}$$

then the power of $I_n \leq 2^0 + 2^1 + 2^2 + \dots + 2^{n-2}$, and the power of $J_n = (n - 1)2^{n-1}$.

That is, the power of $I_n < 2^{n-1}$, and the power of $J_n > 2^n$ for $n \geq 4$. Hence there is a sequence of integers $\{j_n\}_4^\infty$ such that $j_n \neq j_i, i < n, j_n \in J_n$, and $j_n \notin I_n$, since $2^{n-1} + (n - 4) < 2^n, n \geq 4$.

Therefore x_{j_n} may be chosen to be 0 or 1 for any x such that

$$x_{j_n}^0 = (f_k^0(x_0, x_1, \dots, x_{j_n^0-1})) \text{ and } x_{j_n^{1, s_2, \dots, s_n}} = 0.$$

That is, the power of A is c .

(iv) As in Mycielski's paper we define S_M to be the set of all $x = (x_0, x_1, \dots) \in C$ such that for every K_{s_1, s_2, \dots, s_n} of M there is some $i \in K_{s_1, s_2, \dots, s_n}: x_i = 0$. Then the proof of (iv) is obvious.

(v) Obvious from Theorem 10 in Mycielski's paper. Q.E.D.

1.2 THEOREM. *Assuming the continuum hypothesis, the Sierpiński-Erdős theorem is satisfied for Λ and the class of sets of measure zero (first category) in C as defined above.*

PROOF. 1.1 Theorem and [5, p. 76, Theorem 19.6]. Q.E.D.

2. On the Oxtoby-Ulam theorem.

A. *On mixed sigma-ideals.* In 1938 J. C. Oxtoby and S. M. Ulam [6] used to so-called category method to show the existence of a homeomorphism of the unit interval onto itself mapping a given set of first category onto a set of measure zero [5, p. 50]. In this section we shall show the corresponding counterparts of this statement for $\Phi(\Psi)$ (the family of subsets of the unit square defined to be of first category (Lebesgue measure zero) in almost every vertical line in the sense of measure (category) [3, pp. 124–128] employing a similar technique based on an argument provided by the same authors in proving a generalization of the same theorem in 1941 [7, p. 912].

2.1 THEOREM. *There is a homeomorphism of the unit square onto itself mapping a given set in $\Phi(\Psi)$ onto a set of measure zero.*

PROOF. If $K \in \Phi$ then there is a measure zero set S in I such that K_x is of first category for every x in $I - S$. Moreover, K may be represented as the union of two complementary sets: $A = \{(x, y) \in K: x \in I - S\}$ and $B = \{(x, y) \in K: x \in S\}$, such that $K \subset A \cup (S \times I), I^2 - (S \times I)$ (where $I^2 = I \times I$) is dense in I^2 (for $S \times I$ is a set of measure zero) and A_x is of first category for every x .

Now, let $I^2(x) = \{(x, y) \in I^2, \text{ for some fixed } x \in I\}$, then for every $x \in I - S$ there is a sequence $\{P_n^x\}_1^\infty$ of perfect subsets of $I^2(x)$ contained in $I^2 - (A \cup (S \times I))$ whose union is dense in $I^2(x)$ [7, p. 912].

Now, let us consider the countable double sequence $P_n^d, n \in J, d \in D$, where D is any countable dense subset of $I - S$ (this is possible, for $I - S$ is a separable metric space as a subset space of I) of perfect subsets of $I^2(d)$, and hence of I^2 (for $I^2(d)$ is a closed subset of I^2); contained in $I^2 - (A \cup (S \times I))$, and hence in $I^2 - K$; whose union is dense in $D \times I$ and therefore in I^2 (since $D \times I$ is dense in I^2). Therefore we have exhibited a sequence of perfect subsets of I^2 contained in the complement of K whose union is dense in I^2 and established an equivalent condition to this theorem [7, p. 912]. A similar argument for Ψ shall be omitted. Q.E.D.

2.2 THEOREM. *There is a set belonging to both Φ and Ψ that cannot be mapped onto a set of first category by a homeomorphism of the unit square onto itself.*

PROOF. $I^2 = (A \times A) \cup (B \times B) \cup (A \times B) \cup (B \times A)$, where $\{A, B\}$ denotes a partition of I into an F_σ set of first category and a G_δ set of measure zero [5, p. 5]. And hence, $(A \times A) \cup (B \times B) \in \Phi(\Psi)$ and not of first category in I^2 (for $(A \times B) \cup (B \times A)$ the complement of $(A \times A) \cup (B \times B)$ is of first category [5, pp. 54, 57]). Therefore, no homeomorphic image of $(A \times A) \cup (B \times B)$ is of first category in I^2 ; and hence, the corresponding counterpart of the Oxtoby-Ulam theorem fails. Q.E.D.

B. *On the ideals of Schmidt and Mycielski.* Throughout the rest of this section, B shall define the binary sequence of subsets of $[0, 1]$ with the right-hand end points (except 1) excluded; that is, B_n is the set of points in $[0, 1]$ whose n th binary digit is 1, taking the terminating expansion in the case of dyadic rationals, except for 1. More formally, $B_n = \{x: (x_1, x_2, \dots, x_{n-1}, 1, x_{n+1}, \dots) \in [0, 1]: x_i = 0, 1 \text{ and there is no } m \in J: x_i = 1, \text{ for } i > m, \text{ except for } x = 1\}$.

Moreover, $c_B(I)$ shall identify the image of I in C under the Marezewski function of the binary sequence B , given by:

$$c_B(x) = \sum_{n=1}^\infty \frac{2\chi_{B_n}(x)}{3^n}, \quad x \in I,$$

and χ_{B_n} the characteristic function of B_n , for all $B_n \in B$ [9, p. 210].

Note that c_B^{-1} is the restriction of the Cantor function to $c_B(I)$ which turns out to be the Cantor set with the left-hand endpoints of the contiguous intervals removed. Hence, c_B^{-1} is a one-to-one and continuous function from $c_B(I)$ onto I .

Since the Oxtoby-Ulam theorem takes place in I , we shall redefine our σ -ideal Λ as the collection $\zeta = \{c_B^{-1}(S \cap c_B(I)): S \in \Lambda\}$ of subsets of I .

2.3 THEOREM. *ζ is a σ -ideal of subsets of I .*

PROOF. Given a sequence S_n in Λ ,

$$\begin{aligned} \bigcup_n c_B^{-1}(S_n \cap c_B(I)) &= c_B^{-1}\left(\bigcup_n (S_n \cap c_B(I))\right) \\ &= c_B^{-1}\left(\left(\bigcup_n S_n\right) \cap c_B(I)\right) \in \zeta, \end{aligned}$$

for $\bigcup S_n$ is in Λ (Λ being a σ -ideal).

Moreover, if $T \subset c_B^{-1}(S \cap c_B(I))$, for a given set S in Λ , then

$$c_B(T) \subset c_B(c_B^{-1}(S \cap c_B(I))) = S \cap c_B(I) \subset S,$$

for c_B^{-1} is one-to-one. Hence, $c_B(T)$ is in Λ , for S is in Λ (Λ being a σ -ideal).
But

$$T = c_B^{-1}(c_B(T)) = c_B^{-1}(c_B(T) \cap c_B(I))$$

for c_B^{-1} is onto, and hence T is in ζ . Q.E.D.

2.4 THEOREM. *Given S in C such that S is in Λ , then there is a sequence of perfect subsets of C contained in the complement of S whose union is dense in C .*

PROOF. Given S in Λ , and K_{s_1, s_2, \dots, s_n} in M as introduced above we define $P_{s_1, s_2, \dots, s_n} = \{z = (z_0, z_1, \dots) : z_j = 0 \text{ or } 1, \text{ and } z_j = f_j(z_0, z_1, \dots, z_{j-1}) \text{ for } j \text{ in } K_{s_1, s_2, \dots, s_n}, \text{ where } \{f_j\}_{j \in K_{s_1, s_2, \dots, s_n}} \text{ is a winning strategy for player II in game } \Gamma(S, K_{s_1, s_2, \dots, s_n})\}$.

We would like to show $\{P_{s_1, s_2, \dots, s_n}\}_{K_{s_1, s_2, \dots, s_n} \in M}$ is a sequence of perfect subsets of C contained in the complement of S whose union is dense in C .

(a) P_{s_1, s_2, \dots, s_n} is closed.

PROOF OF (a). If x is in the complement of P_{s_1, s_2, \dots, s_n} , then $x_j \neq f_j(x_0, x_1, \dots, x_{j-1})$ for some j in K_{s_1, s_2, \dots, s_n} . Moreover, if $N_x = \{y \in I : y_k = x_k, \text{ for all } k \leq j\}$, then N_x defines a neighborhood of x contained in the complement of P_{s_1, s_2, \dots, s_n} . Hence, the complement of P_{s_1, s_2, \dots, s_n} is open.

(b) P_{s_1, s_2, \dots, s_n} is dense in itself.

PROOF OF (b). Suppose x is in P_{s_1, s_2, \dots, s_n} , i.e., $x_j = f_j(x_0, x_1, \dots, x_{j-1})$ for all j in K_{s_1, s_2, \dots, s_n} , and hence for j in K_{s_1, s_2, \dots, s_n} such that $j \leq m$, for any m .

Now, given $\epsilon > 0$, choose y to agree with x for finitely and sufficiently many entries (say, $y_j = x_j$ for $j \leq m$) such that $d(x, y) < \epsilon$. On the other hand, for $j > m$ choose y_j such that $y_j = f_j(y_0, y_1, \dots, y_{j-1})$ for j in K_{s_1, s_2, \dots, s_n} , $y_k \neq x_k$ for k equal to the first integer $> m$: $k \notin K_{s_1, s_2, \dots, s_n}$, and $y_k = x_k$ for all other $k \notin K_{s_1, s_2, \dots, s_n}$. This is possible for K_{s_1, s_2, \dots, s_n} cannot include every integer from some point on, otherwise $K_{s_1, s_2, \dots, s_n-1, 0}$ or 1 were to finite. A contradiction.

Hence $y = (y_0, y_1, \dots, y_j, \dots) : y_j = f_j(y_0, y_1, \dots, y_{j-1})$ for all $j \in K_{s_1, s_2, \dots, s_n}$, i.e., $y \in P_{s_1, s_2, \dots, s_n}$ and $x \neq y$.

Therefore, if x is in P_{s_1, s_2, \dots, s_n} , then x is a limit point of P_{s_1, s_2, \dots, s_n} .

(c)

$$\bigcup_{s_1, s_2, \dots, s_n: K_{s_1, s_2, \dots, s_n} \in M} P_{s_1, s_2, \dots, s_n}$$

is dense in C .

PROOF OF (c). Let $x \in C$. Given $\epsilon > 0$, choose y to agree with x for finitely and sufficiently many entries (say, $y_j = x_j$ for $j \leq m$) such that $d(x, y) < \epsilon$.

Now consider $s_1, s_2, \dots, s_n: j \in K_{s_1, s_2, \dots, s_n} \Rightarrow j > m$. This is possible, since $K_{s_1, s_2, \dots, s_{n+1}} \subset K_{s_1, s_2, \dots, s_n}$, and $K_{s_1, s_2, \dots, s_{n-1}, 0} \cap K_{s_1, s_2, \dots, s_{n-1}, 1} = \emptyset$.

Now define $y: y_j = f_j(y_0, y_1, \dots, y_{j-1})$, if $j \in K_{s_1, s_2, \dots, s_n}$ (and note that $j > m$), and $y_j = x_j$, for all $j \notin K_{s_1, s_2, \dots, s_n}$.

Then

$$y \in P_{s_1, s_2, \dots, s_n} \Rightarrow y \in \bigcup P_{s_1, s_2, \dots, s_n}$$

Therefore, $x \in I$, hence $x \in \overline{\bigcup P_{s_1, s_2, \dots, s_n}}$.

(d) P_{s_1, s_2, \dots, s_n} is contained in the complement of S .

PROOF OF (d). If z is in P_{s_1, s_2, \dots, s_n} , then $z_j = f_j(z_0, z_1, \dots, z_{j-1})$, $j \in K_{s_1, s_2, \dots, s_n}$ and hence $z = (z_0, z_1, \dots)$ is in the complement of S . Q.E.D.

Whether an analogue of the Oxtoby-Ulam theory holds for Λ , that is, whether for each $S \in \Lambda$ there exists an automorphism h of C such that $h(S)$ has Haar measure zero, remains an open question. However, such a theorem holds for ζ .

2.5 THEOREM. For each $S \in \zeta$ there is a homeomorphism h of I onto I such that $h(S)$ has Lebesgue measure zero.

PROOF. $c_B(S)$ belongs to Λ , hence by 2.4 Theorem there is a sequence $\{P_i\}$ of perfect sets contained in $C - c_B(S)$ whose union is dense in C . Because $C - c_B(I)$ is countable, each of the sets $P_i \cap c_B(I)$ contains a sequence $\{P_{i,j}\}, j \in J$, of compact and perfect sets whose union is dense in P_i . Hence $\{P_{i,j}\}$ is a countable family of compact and perfect subsets of $c_B(I) - c_B(S)$ whose union is dense in C . Because c_B^{-1} is one to one and continuous on $c_B(I)$, it follows that $\{c_B^{-1}(P_{i,j})\}$ is a countable family of compact and perfect subsets of $I - S$ whose union is dense in I . The conclusion then follows from [7, p. 912]. Q.E.D.

REFERENCES

1. P. R. Halmos, *Measure theory*, Van Nostrand, Princeton, N.J., 1950.
2. K. Kuratowski, *Topology*, Vol. I, Academic Press, New York, 1966.
3. C. G. Mendez, *On sigma-ideals of sets*, Proc. Amer. Math. Soc. **60** (1976), 124–128.
4. J. Mycielski, *Some new ideals of sets on the real line*, Colloq. Math. **20** (1969), 71–76.
5. J. C. Oxtoby, *Measure and category*, Springer-Verlag, New York, 1971.
6. J. C. Oxtoby and S. M. Ulam, *On the equivalence of any set of first category to a set of measure zero*, Fund. Math. **31** (1938), 201–206.

7. _____, *Measure-preserving homeomorphisms and metrical transitivity*, Ann. of Math. (2) **43** (1941), 874–920.

8. W. Schmidt, *On badly approximable numbers and certain games*, Trans. Amer. Math. Soc. **123** (1966), 178–199.

9. E. Szpilrajn, *The characteristic function of a sequence of sets and some of its applications*, Fund. Math. **31** (1938), 207–225.

DEPARTMENT OF MATHEMATICAL SCIENCES, METROPOLITAN STATE COLLEGE, DENVER,
COLORADO 80204