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k-REGULAR EMBEDDINGS OF THE PLANE

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ABSTRACT. A map $f: X \to R^n$ is said to be k-regular if whenever $x_1, \ldots, x_k$ are distinct points of X, then $f(x_1), \ldots, f(x_k)$ are linearly independent. Such maps are of interest in the theory of Čebyšev approximation. In this paper, configuration spaces and homological methods are used to show that there does not exist a k-regular map of R^2 into $R^{2k-\alpha(k)-1}$ where $\alpha(k)$ denotes the number of ones in the dyadic expansion of k. This result is best possible when k is a power of 2.

1. Introduction. Let $k \leqslant n$ be positive integers. A continuous map f: $X \to R^n$ is *k-regular* if whenever $x_1, \ldots, x_k$ are distinct points of X, then $f(x_1), \ldots, f(x_k)$ are linearly independent.

EXAMPLE 1.1. f: $R \to R^k$ given by $f(t) = (1, t, t^2, \ldots, t^{k-1})$ is k-regular, as is seen by the nonvanishing of the Vandermonde determinant.

EXAMPLE 1.2. f: $R^2 = C \to R^{2k-1} = R \times C^{k-1}$ given by $f(z) = (1, z, z^2, \ldots, z^{k-1})$ is k-regular.

k-regular maps are of relevance in the theory of Čebyšev approximation. The connection is as follows: Let X be a compact subspace of some Euclidean space, and suppose $f_1, \ldots, f_n$ are continuous, linearly independent real-valued functions defined on X. For an arbitrary continuous g: $X \to R$, let $\mathfrak{B}(g)$ denote the set of best approximations to g by linear combinations of $f_1, \ldots, f_n$, i.e. if F denotes the n-dimensional vector space of functions spanned by the f_i and $m_g = \inf_{f \in F} \sup_{x \in X}|g(x) - f(x)|$, then $\mathfrak{B}(g) = \{f \in F$: $\sup_{x \in X}|g(x) - f(x)| = m_g\}$. Then the following theorem holds:

THEOREM (HAAR-KOLMOGOROV-RUBINSTEIN). *For every continuous g: $X \to R$, the dimension of the set $\mathfrak{B}(g)$ is $\leqslant n - k$ if and only if $(f_1, \ldots, f_n)$: $X \to R^n$ is k-regular.*

For a proof see [**9**, pp. 237–242].

For example, $f = (f_1, \ldots, f_n)$: $X \to R^n$ is n-regular if and only if $f_1, \ldots, f_n$ form a Haar system on X, i.e. $\det(f_i(x_j)) \neq 0$ whenever $x_1, \ldots, x_n$ are distinct points of X. This is the case if and only if every continuous g: $X \to R$ has a unique best approximation by linear combinations of the f_i.

Work on existence and nonexistence of k-regular maps by nonalgebraic

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topological methods appears in **[1]**, **[2]**, and **[6]**. For example, in **[1]**, the following is proved:

THEOREM 1.3 (BOLTJANSKIĬ-RYŠKOV-ŠAŠKIN). *If a* $2k$*-regular map of* R^n *into* R^N *exists, then* $N \geqslant (n + 1)k$.

In fact, if f: $R^n \to R^N$ is $2k$-regular, then if $E_1, \ldots, E_k$ are pairwise-disjoint n-discs in R^n, the map g: $E_1 \times \cdots \times E_k \times (R - \{0\})^k \to R^N$ given by $g(x_1, \ldots, x_k; t_1, \ldots, t_k) = \Sigma_i t_i f(x_i)$ is injective, and so N cannot be less than the dimension of the space on the left.

In **[5]**, homological methods, using configuration spaces, are used to get a nonexistence result. The present paper uses a variant of the method of **[5]**, together with results on the cohomology of configuration spaces of R^n obtained in **[3]**, to prove the following:

THEOREM 1.4. *There does not exist a* k*-regular map of* R^2 *into* $R^{2k-\alpha(k)-1}$ *where* $\alpha(k)$ *denotes the number of ones in the dyadic expansion of* k.

Note that 1.4 gives an improvement over 1.3 in case $n = 2$. When k is a power of 2, Example 1.2 shows that 1.4 is best possible.

2. Equivariant maps from configuration spaces to Stiefel manifolds. If X is a topological space, let $F(X, k)$ denote the kth configuration space of X, i.e. the subspace of X^k consisting of all ordered k-tuples of distinct points in X. Let $V_k(R^N)$ denote the Stiefel manifold of linearly independent (not necessarily orthonormal) k-frames in R^N. The symmetric group Σ_k acts freely on $F(X, k)$ and $V_k(R^N)$ by permuting factors: $\sigma(y_1, \ldots, y_k) = (y_{\sigma^{-1}(1)}, \ldots, y_{\sigma^{-1}(k)})$, $\sigma \in \Sigma_k$, $(y_1, \ldots, y_k) \in F(X, k)$ or $V_k(R^N)$. A k-regular map f: $X \to R^N$ yields a Σ_k-equivariant map g: $F(X, k) \to V_k(R^N)$ given by $g(x_1, \ldots, x_k) = (f(x_1), \ldots, f(x_k))$.

Σ_k acts orthogonally on R^k by permuting factors. Thus we obtain real k-plane bundles

$$F(X, k) \times_{\Sigma_k} R^k \to F(X, k)/\Sigma_k$$

and

$$V_k(R^N) \times_{\Sigma_k} R^k \to V_k(R^N)/\Sigma_k.$$

PROPOSITION 2.1. *There exists a* Σ_k*-equivariant map* $F(X, k) \to V_k(R^N)$ *if and only if the* k*-plane bundle* $F(X, k) \times_{\Sigma_k} R^k \to F(X, k)/\Sigma_k$ *admits an* $N - k$*-plane inverse.*

PROOF. If such an $N - k$-plane inverse existed, there would exist a map f: $F(X, k) \times_{\Sigma_k} R^k \to R^N$ whose restriction to each fibre is an R-monomorphism. Then g: $F(X, k) \to V_k(R^N)$ given by $g(x) = (f(x, e_1), \ldots, f(x, e_k))$, where $e_1, \ldots, e_k$ is the standard basis of R^k, is Σ_k-equivariant.

The converse follows from the fact that $V_k(R^N) \times_{\Sigma_k} R^k \to V_k(R^N)/\Sigma_k$

admits an $N - k$-plane inverse. In fact, $f\colon V_k(R^N) \times_{\Sigma_k} R^k \to R^N$ given by $f(y_1, \ldots, y_k; t_1, \ldots, t_k) = \Sigma_i t_i y_i$ is well defined and its restriction to each fibre is an R-monomorphism.

COROLLARY 2.2. *If a k-regular map $X \to R^N$ exists, then $F(X, k) \times_{\Sigma_k} R^k \to F(X, k)/\Sigma_k$ admits an $N - k$-plane inverse.*

COROLLARY 2.3. *If X admits a Haar system $f_1, \ldots, f_n$, then the bundle $F(X, n) \times_{\Sigma_n} R^n \to F(X, n)/\Sigma_n$ is trivial as an R^n-bundle with group Σ_n.*

3. Proof of Theorem 1.4. Write $P_{2,k}$ for the vector bundle $F(R^2, k) \times_{\Sigma_k} R^k \to F(R^2, k)/\Sigma_k$. By [**4**, Theorem 1], the Whitney sum of two copies of $P_{2,k}$ is trivial, and so $\bar{w}_i(P_{2,k}) = w_i(P_{2,k})$ for all i. Thus 1.4 will follow from 2.2 and the following:

THEOREM 3.1. $w_{k-\alpha(k)}(P_{2,k}) \neq 0$.

LEMMA 3.2. *Let k be a power of 2. Then $w_{k-1}(P_{2,k}) \neq 0$.*

PROOF OF 3.2. All homology and cohomology groups are with $Z/2Z$ coefficients. Let $\rho\colon B\Sigma_k \to BO(k)$ be induced by the regular representation $\Sigma_k \to O(k)$. The following composite is a classifying map for $P_{2,k}$:

$$F(R^2, k)/\Sigma_k \xrightarrow{\sigma_k} F(R^\infty, k)/\Sigma_k \cong B\Sigma_k \xrightarrow{\rho} BO(k)$$

where $R^\infty = \text{inj lim}_n R^n$ and $\sigma_k = \text{inj lim}_n \sigma_{n,k}$ where $\sigma_{n,k}$ is given in [**7**, p. 35].

Let $\mathcal{C}_n$ denote the little n-cubes operad with $C_n X$ the associated construction given by J. P. May [**7**, p. 13] $\mathcal{C}_n(j)$ is equivariantly homotopically equivalent to $F(R^n, j)$ and by [**7**, p. 36], the following diagram commutes:

$$\begin{array}{lcl} \mathcal{C}_2(k)/\Sigma_k & \xrightarrow{\sigma_k} & \mathcal{C}_\infty(k)/\Sigma_k \\ \downarrow g_2 & & \downarrow g \\ F(R^2, k)/\Sigma_k & \xrightarrow{\sigma_k} & F(R^\infty, k)/\Sigma_k \end{array}$$

where the top $\sigma_k = \text{inj lim}_n \sigma_{n,k}$, g_2 is a homotopy equivalence, and $g = \text{inj lim}_n g_n$ is a homotopy equivalence. Note that $C_n S^0 = \amalg_{j \geqslant 0} \mathcal{C}_n(j)/\Sigma_j$ and that the map σ_k, on the level of little cubes, is precisely the map σ: $C_2 S^0 \to C_\infty S^0$, $\sigma = \text{inj lim}_n \sigma_n$ (by [**7**, Theorem 5.2]), restricted to the kth component. We use the observations to compute σ_k in homology.

Let [**1**] denote the element in $H_0(S^0)$ represented by the non-base-point. By [**3**, §3], $H_*(C_n S^0)$ is given in terms of Dyer-Lashof operations on [**1**].

Let $k = 2^i$ and consider the sequence $I = (2^{i-1}, 2^{i-2}, \ldots, 2, 1)$. By [**3**, §§1,4], the element $Q^I[1]$ is defined in $H_*(C_2 S^0)$ and by the filtration arguments given there, we see that $Q^I[1] \in H_*(\mathcal{C}_2(k)/\Sigma_k)$. Since σ is a morphism of $\mathcal{C}_2$-spaces, we have the formula $\sigma_{k*} Q^I[1] = Q^I[1]$.

By [**8**, Theorem 4.7], $\langle \rho^* w_{k-1}, Q^I[1] \rangle = 1$ where $\langle\ ,\ \rangle$ denotes the Kronecker index. Hence

$$\begin{aligned}\langle \sigma_k^* \rho^* w_{k-1}, Q^I[1] \rangle &= \langle \rho^* w_{k-1}, \sigma_{k*} Q^I[1] \rangle \\ &= \langle \rho^* w_{k-1}, Q^I[1] \rangle = 1\end{aligned}$$

and so $w_{k-1}(P_{2,k}) = \sigma_k^* \rho^* w_{k-1} \neq 0$.

PROOF OF 3.1. Write $k = \sum_{i=1}^{\alpha(k)} j(i)$ where $j(i) = 2^{n(i)}$, $n(1) < n(2) < \cdots < n(\alpha(k))$. We have a map of k-plane bundles f: $P_{2,j(1)} \times \cdots \times P_{2,j(\alpha(k))} \to P_{2,k}$ as follows: Choose pairwise-disjoint open discs $E_1, \ldots, E_{\alpha(k)}$ in R^2. Then we can regard $P_{2,j(i)}$ as

$$F(E_i, j(i)) \times_{\Sigma_{j(i)}} R^{j(i)} \to F(E_i, j(i))/\Sigma_{j(i)}.$$

Define f by

$$f((x_1, v_1), \ldots, (x_{\alpha(k)}, v_{\alpha(k)})) = (x_1, \ldots, x_{\alpha(k)}; v_1, \ldots, v_{\alpha(k)})$$
$$(x_i, v_i) \in F(E_i, j(i)) \times_{\Sigma_{j(i)}} R^{j(i)}.$$

Note that for any j, $w_j(P_{2,j}) = 0$ since $P_{2,j}$ admits the nowhere-zero section $(x) \mapsto (x; 1, \ldots, 1)$. Hence

$$\begin{aligned} f^* w_{k-\alpha(k)}(P_{2,k}) &= w_{k-\alpha(k)}(P_{2,j(1)} \times \cdots \times P_{2,j(\alpha(k))}) \\ &= w_{j(1)-1}(P_{2,j(1)}) \times \cdots \times w_{j(\alpha(k))-1}(P_{2,j(\alpha(k))}) \neq 0, \end{aligned}$$

completing the proof.

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