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SYMMETRIC GRAPHS WITH PROJECTIVE SUBCONSTITUENTS

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ABSTRACT. Let Γ be a finite, undirected, connected graph and G a subgroup of $\mathrm{aut}(\Gamma)$ acting transitively on the vertex set of Γ such that the stabilizer $G(x)$ in G of a vertex x contains a normal subgroup which induces a permutation group on the set of vertices adjacent to x isomorphic to $PSL(n, q)$ with $n \geqslant 3$. A bound for $|G(x)|$ depending only on n and q is shown to exist under certain conditions.

Let Γ be an undirected graph and G a subgroup of $\mathrm{aut}(\Gamma)$ acting transitively on the vertex set of Γ. Let x be an arbitrary vertex of Γ. We denote by $\Gamma(x)$ the set of vertices adjacent to x and by $G(x)^{\Gamma(x)}$ the permutation group induced by the stabilizer $G(x)$ of x in G on $\Gamma(x)$; $G(x)^{\Gamma(x)}$ is called the subconstituent of the pair (Γ, G). For each $i \in \mathbf{N}$ let $G_i(x) = \{a \in G(x) | a \in G(y) \text{ if } \partial(x, y) \leqslant i\}$ where $\partial(x, y)$ denotes the distance between x and y. Let $s \in \mathbf{N}$. An s-path is a sequence $(x_0, x_1, \ldots, x_s)$ of $s + 1$ vertices x_i such that $x_{i-1} \in \Gamma(x_i)$ if $1 \leqslant i \leqslant s$ and $x_{i-2} \neq x_i$ if $2 \leqslant i \leqslant s$. For each s-path $(x_0, \ldots, x_s)$ let $G(x_0, \ldots, x_s) = G(x_0) \cap \cdots \cap G(x_s)$ and $G_i(x_0, \ldots, x_s) = G_i(x_0) \cap \cdots \cap G_i(x_s)$ for each $i \in \mathbf{N}$. The graph Γ is called (G, s)-transitive if G acts transitively on the set of s-paths but not on the set of $(s + 1)$-paths in Γ.

In **[6]**, Tutte showed that if Γ is finite, connected, trivalent and (G, s)-transitive with $s \geqslant 1$ then $s \leqslant 5$ and $|G(x)| = 3 \cdot 2^{s-1}$. Subsequently considerable attention has been given to the problem of finding a similar result for graphs of arbitrary valency; see, for instance, **[1]**, **[3]** and **[9]**. In the present paper we discuss the case that $PSL(n, q) \lneqq G(x)^{\Gamma(x)} \lneqq P\Gamma L(n, q)$ with $n \geqslant 3$. In **[8]**, it was shown that $s \leqslant 3$ in this case; however, it seems much more difficult to find a bound for $|G(x)|$ when $PSL(n, q) \lneqq G(x)^{\Gamma(x)} \lneqq P\Gamma L(n, q)$ with $n \geqslant 3$. We prove the following partial result:

THEOREM. *Let n be a natural number greater than two and q a power of some prime p. Let Γ be a finite, undirected, connected graph and suppose that* $\mathrm{aut}(\Gamma)$ *contains a subgroup G such that Γ is $(G, 3)$-transitive with $PSL(n, q) \lneqq G(x)^{\Gamma(x)} \lneqq P\Gamma L(n, q)$ for every vertex x (where $PSL(n, q)$ and $P\Gamma L(n, q)$ are to be considered as acting on the set of points of the projective space $PG(n - 1, q)$ in the usual fashion). Then $|G_1(x, y)| \,\big|\, q^{(n-1)^2}$ for every edge $\{x, y\}$ if $p \geqslant 5$.*

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Before turning to the proof, we note two important examples. First, let q be an arbitrary prime power, $n \geqslant 3$, G the group of all collineations and correlations of $PG(2n-1, q)$ and Γ the bipartite graph whose vertices are the $(n-2)$- and $(n-1)$-dimensional subspaces of $PG(2n-1, q)$ where two such subspaces are defined to be adjacent when one of them contains the other. Let $\{x, y\}$ be an edge of Γ. Then Γ is connected, $(G, 3)$-transitive, $G(x)^{\Gamma(x)} = P\Gamma L(n, q)$ and $|G_1(x, y)| = q^{(n-1)^2}$. Thus the bound for $|G_1(x, y)|$ given in the theorem is the best possible one.

When q is even, $|G_1(x, y)| \,\big|\, q^{(n-1)^2}$ need not hold, at least when $n = 3$. To see this, let S be the metasymplectic space associated with the group $F_4(q)$, q even (see [**5**, (10.13)]). Let Γ be the bipartite graph whose vertices are the lines and planes of S where a line L is defined to be adjacent to a plane P when P contains L. Let G be the group of all collineations and correlations of S. By [**5**, (5.10)], G acts transitively on the vertex set of Γ. For each plane P of S, $G(P)$ acts like $P\Gamma L(3, q)$ on the projective space consisting of P together with the subspaces it contains. If $\{x, y\}$ is an edge of Γ, then $O_2(G(x, y))$ is isomorphic to a 2-Sylow group of $F_4(q)$ and thus $|G_1(x, y)| = q^{18}$.

To show that Γ is $(G, 3)$-transitive, let P_1 and P_2 be planes of S and L_1 and L_2 lines such that (P_1, L_1, P_2, L_2) is a 3-path in Γ. Since L_1 and L_2 both lie in P_2, $L_1 \cap L_2 \neq \varnothing$; let α be the point of S lying in $L_1 \cap L_2$. Since $P_1 \cap P_2$ contains L_1, there exists dually a symplectum s of S containing both P_1 and P_2. Let Δ be the flag complex of S. Let $C_0 = \{\alpha, L_1, P_1, s\}$, $C_1 = \{\alpha, L_1, P_2, s\}$ and $C_2 = \{\alpha, L_2, P_2, s\}$. Then C_0, C_1 and C_2 are chambers of Δ and (C_0, C_1, C_2) is a minimal gallery. Since Δ is a building, there is an apartment Σ of Δ containing both C_0 and C_2; by [**5**, (3.18)], Σ contains C_1 too. If Π is the subgraph of Γ induced by the lines and planes of S in Σ, then (P_1, L_1, P_2, L_2) is a 3-path in Π. Let Φ be the bipartite graph whose vertices are the edges and triangles of the regular polytope $\{3, 4, 3\}$ (see [**2**, p. 149]) where an edge E is defined to be adjacent to a triangle T when T contains E. Then $\Pi \cong \Phi$ and the stabilizer of Σ in G induces an automorphism group on Π isomorphic to the symmetry group $[3, 4, 3]$ of $\{3, 4, 3\}$. For each edge E of $\{3, 4, 3\}$, the stabilizer of E in $[3, 4, 3]$ acts transitively on the set of twelve 3-paths of Φ beginning at E. Since G acts transitively on the set of all apartments of Δ, Γ is in fact $(G, 3)$-transitive.

In the following lemma we list a few elementary properties of the projective groups which we will need in the proof of the theorem; verification is left to the reader:

LEMMA. *Let H be a subgroup of $P\Gamma L(n, q)$ (with $n \geqslant 3$) containing $PSL(n, q)$ considered as acting in the usual fashion on $PG(n-1, q)$, q a power of a prime $p \geqslant 5$. Let X be the set of points of $PG(n-1, q)$, $x \in X$ arbitrary and Y the set of lines passing through x.*

(i) *If $1 \neq M \trianglelefteq K \trianglelefteq\trianglelefteq H(x)$, where $H(x)$ denotes the stabilizer of x in H, and $K^Y \neq 1$ then $K^Y \geqslant PSL(n-1, q)$ and $M \geqslant O_p(H(x))$. $|O_p(H(x))| =$*

q^{n-1}, $O_p(H(x))^Y = 1$ *and* $O_p(H(x))$ *acts transitively on* $g - \{x\}$ *for every* $g \in Y$. *The order of the largest subgroup of* $H(x)$ *acting trivially on* Y *divides* $(q-1)q^{n-1}$.

(ii) *If* $\{x_1, \ldots, x_n\}$ *is a frame of* $PG(n-1, q)$ (*i.e., a set of n points no three of which are collinear*), *then* $\langle O_p(H(x_1)), \ldots, O_p(H(x_n))\rangle$ *acts transitively on* X.

We now begin the proof of the theorem. Let Γ be an arbitrary graph fulfilling the hypotheses. By [**7**, (7.61)] there exist two vertices u and v with $\partial(u, v) = 2$. If $|\Gamma(u) \cap \Gamma(v)| \geqslant 2$ then Γ is a complete bipartite graph since G acts transitively on the set of all 3-paths in Γ. Thus we may assume that $|\Gamma(u) \cap \Gamma(v)| = 1$ whenever $\partial(u, v) = 2$.

For every vertex x of Γ we denote by $\mathcal{P}(x)$ the projective space of dimension $n-1$ with point set $\Gamma(x)$ induced by $G(x)$. For each edge $\{x, y\}$ let $\mathcal{P}(x, y)$ be the projective space of dimension $n-2$ whose subspaces of dimension m are the subspaces of $\mathcal{P}(x)$ of dimension $m+1$ containing y $(0 \leqslant m \leqslant n-2)$. Let $P(x, y)$ be the set of points of $\mathcal{P}(x, y)$. For each element a of $G(x, y)$ let a_{xy} be the permutation that a induces on $P(x, y)$.

Now let $\{x, y\}$ be an arbitrary edge of Γ. We claim that $G_1(x)^{P(y,x)} \trianglerighteq PSL(n-1, q)$. Suppose not. By the lemma, $G_1(x)^{P(y,x)} = 1$ since $G_1(x) \trianglelefteq G(x, y)$. We define a map f: $G(x, y)^{P(x,y)} \to G(x, y)^{P(y,x)}$ as follows: Given an arbitrary element $c \in G(x, y)^{P(x,y)}$, pick an element $a \in G(x, y)$ with $a_{xy} = c$ and set $f(c) = a_{yx}$. Suppose that b is a second element of $G(x, y)$ with $b_{xy} = c$. Let $K = \{d \in G(x, y) | d_{xy} = 1\}$. If $a_{yx} \neq b_{yx}$ then K acts nontrivially on $P(y, x)$ and thus, by the lemma, $K^{P(y,x)} \trianglerighteq PSL(n-1, q)$ since $K \trianglelefteq G(x, y)$. Since $[K: G_1(x)]$, by the lemma, divides $(q-1)q^{n-1}$ but $|PSL(n-1, q)|$ does not, $G_1(x)$ acts nontrivially on $P(y, x)$. This contradicts the assumption. Thus f is well defined. Since f is clearly a homomorphism and invertible, f is induced, according to [**4**], by a map φ from $\mathcal{P}(x, y)$ to $\mathcal{P}(y, x)$ which is either a correlation or a collineation. Let $w \in \Gamma(x) - \{y\}$ and $\alpha \in P(x, y)$ with $w \in \alpha$ be arbitrary; we have $G(w, x, y) \leqslant G(\alpha, x, y)$ where $G(\alpha, x, y)$ denotes the stabilizer of α in $G(x, y)$. Let $\beta = \varphi(\alpha)$. β is a subspace of $\mathcal{P}(y, x)$ of dimension 0 or $n-3$. Since $G(\alpha, x, y) = G(x, y, \beta)$, $G(w, x, y)$ acts intransitively on $\Gamma(y) - \{x\}$. This contradicts the hypothesis that G acts transitively on the set of all 3-paths in Γ. We conclude that $G_1(x)^{P(y,x)} \trianglerighteq PSL(n-1, q)$, as claimed.

Let z be an arbitrary vertex in $\Gamma(y) - \{x\}$. According to [**3**, (2.3)], $|G_1(x, y)|$ is a prime power. Since $G_1(x, y) \trianglelefteq G_1(y) \trianglelefteq G(y, z)$, $G_1(x, y)^{P(z,y)} = 1$ and $|G_1(x, y)^{\Gamma(z)}| = 1$ or q^{n-1} by the lemma. If $G_1(x, y)^{\Gamma(z)} = 1$ then $G_1(x, y) = G_1(y, z)$ and thus $G_1(x, y) \trianglelefteq \langle G(x, y), G(y, z)\rangle = G(y)$. Let d be an element of G exchanging x and y. Then $G_1(x, y) \trianglelefteq \langle G(y), d\rangle$. Since Γ is connected, $\langle G(y), d\rangle = G$ and thus $G_1(x, y) = 1$. Hence we may assume that $|G_1(x, y)^{\Gamma(z)}| = q^{n-1}$. In particular, $G_1(x, y)$ is a p-group; since $G_1(x, y) \trianglelefteq G_1(x) \trianglelefteq G(x)$, we have $G_1(x, y) \leqslant O_p(G(x))$ and thus $O_p(G(x)) \neq 1$. We define a map f: $G_1(y)^{P(x,y)} \to G_1(y)^{P(z,y)}$ as follows: Given an arbitrary element $c \in G_1(y)^{P(x,y)}$, pick an element $a \in G_1(y)$ with $a_{xy} = c$ and set $f(c) = a_{zy}$. It is easily seen, just as in the previous paragraph, that f is well defined.

Suppose that f is induced by a correlation from $\mathcal{P}(x,y)$ to $\mathcal{P}(z,y)$. Let $u \in \Gamma(y) - \{x\}$, $u \neq z$, and $\alpha \in P(x,y)$ be arbitrary. Then there exist copoints A of $\mathcal{P}(z,y)$ and B of $\mathcal{P}(u,y)$ such that $G_1(y) \cap G(x,\alpha) = G_1(y) \cap G(z,A) = G_1(y) \cap G(u,B)$ where $G(x,\alpha)$ denotes the stabilizer of α in $G(x)$, etc. Similarly, however, there exists a point β of $\mathcal{P}(u,y)$ such that $G_1(y) \cap G(z,A) = G_1(y) \cap G(u,\beta)$. Thus $G_1(y) \cap G(u,B) = G_1(y) \cap G(u,\beta)$ although $G_1(y)^{P(u,y)} \gtrsim PSL(n-1,q)$. With this contradiction we conclude that f is induced by a collineation from $\mathcal{P}(x,y)$ to $\mathcal{P}(z,y)$.

Now let v be an arbitrary vertex of Γ and g an arbitrary line of $\mathcal{P}(v)$. For each pair of points u and w in $\Gamma(v)$ there is a collineation φ_{uw}: $\mathcal{P}(u,v) \to \mathcal{P}(w,v)$ such that $G_1(v) \cap G(u,\alpha) = G_1(v) \cap G(w,\varphi_{uw}(\alpha))$ for every point α of $\mathcal{P}(u,v)$. For each point u of g choose a point α_u of $\mathcal{P}(u,v)$ in such a way that $\varphi_{uw}(\alpha_u) = \alpha_w$ for every two points u and w of g. Let $\pi = \{\alpha_u | u \in g\}$ and let $d(v,g,\pi)$ denote the subgraph of Γ induced by v together with the $q^2 + 2q + 1$ vertices which are points of g or one of the α_u in π. The vertex v will be called the center of $d(v,g,\pi)$. Let D be the set of all such subgraphs $d(v,g,\pi)$, v an arbitrary vertex of Γ, g an arbitrary line of $\mathcal{P}(v)$ and π an arbitrary set $\{\alpha_u \in P(u,v) | u \in g\}$ fulfilling the above condition. Let Δ be the undirected graph with vertex set D where two vertices $d(v_1,g_1,\pi_1)$ and $d(v_2,g_2,\pi_2)$ are defined to be adjacent when $v_1 \in g_2$, $v_2 \in g_1$, $g_1 \in \pi_2$ and $g_2 \in \pi_1$. The graph Δ is regular of valency $q+1$ and G acts faithfully as a group of automorphisms of Δ.

Note that each 3-path (w,x,y,z) in Γ determines a unique vertex $d(x,g,\pi)$ of Δ with center x such that w and $y \in g$ and $z \in \alpha_y \in \pi$; let d_{wxyz} denote this vertex. The group $G_1(y) \cap G(x,g)$ stabilizes d_{wxyz} and acts transitively on $g - \{y\}$. It follows that $PSL(2,q) \lesssim G(d)^{\Delta(d)} \lesssim P\Gamma L(2,q)$ for every $d \in D$. Since G acts transitively on the set of 3-paths of Γ, G acts transitively on D. Thus there exists a $t \geq 2$ such that the graph Δ is (G,t)-transitive. In **[10]**, it was shown that if Δ is an arbitrary finite (G,t)-transitive graph with $PSL(2,q) \lesssim G(d)^{\Delta(d)} \lesssim P\Gamma L(2,q)$ for every vertex d, then either $t \leq 4$ or $t = 2p+1$ with $2 \leq p \leq 3$.

Now let $\{x_2, x_3\}$ be an arbitrary edge of Γ. If $G_2(x_2) = G_2(x_3)$, then $G_2(x_3) \trianglelefteq \langle G(x_2), G(x_3)\rangle$; since Γ is connected, $\langle G(x_2), G(x_3)\rangle$ acts transitively on the edge set of Γ and so $G_2(x_3) = 1$. Suppose that $G_2(x_2) \neq G_2(x_3)$. Then there exists a vertex $x_1 \in \Gamma(x_2)$ such that $G_2(x_3) \not\leq G_1(x_1)$. There exist vertices x_0, x_4, x_5 and x_6 such that $(x_0, x_1, x_2, \ldots, x_6)$ is a 6-path in Γ and $(d_{0123}, d_{1234}, d_{2345}, d_{3456}, d_{6543})$, where $d_{ijkl} = d_{x_i x_j x_k x_l}$, is a 4-path in Δ which we denote by W. Let α be the point of $\mathcal{P}(x_1,x_2)$ containing x_0. We have $G_2(x_3) \leq G(W)$ since $G_2(x_3)^{P(x_1,x_2)} \leq G_1(x_2,x_3)^{P(x_1,x_2)} = 1$ and $G_2(x_3)^{P(x_5,x_4)}$

$\leqslant G_1(x_3, x_4)^{P(x_5, x_4)} = 1$. Since $G_2(x_3) \trianglelefteq G_1(x_2) \trianglelefteq G(x_1, x_2)$, $G_2(x_3)$ acts transitively on $\alpha - \{x_2\}$ by the lemma. Thus $G(W)$ acts transitively on $\Delta(d_{0123}) - \{d_{1234}\}$. Since $t \geqslant 2$, G contains an element mapping an arbitrary 2-path in Δ to $(d_{2345}, d_{1234}, d_{0123})$; thus $t \geqslant 3$. Hence G contains an element mapping an arbitrary 3-path to $(d_{3456}, d_{2345}, d_{1234}, d_{0123})$; thus $t \geqslant 4$ and so G contains an element mapping an arbitrary 4-path to W. Hence $t \geqslant 5$. Since $p \geqslant 5$ implies that $t \leqslant 4$, we conclude that $G_2(x_3) = 1$ after all.

Finally, let x be an arbitrary vertex and $\{x_1, \ldots, x_n\}$ a frame of $\mathcal{P}(x)$. Let $J = G_1(x_1) \cap \cdots \cap G_1(x_n)$. Since $|G_1(x_1, x)^{\Gamma(x_i)}| = q^{n-1}$ for $2 \leqslant i \leqslant n$, we have $|J| \geqslant |G_1(x_1, x)|/q^{(n-1)^2}$. For $1 \leqslant i \leqslant n$ let $Q(i)$ denote the center of $O_p(G(x_i))$. If $Q(i) \leqslant G_1(x)$, then $Q(i) \leqslant G_2(x_i) = 1$ since $Q(i) \trianglelefteq G(x_i)$. Since $O_p(G(x_i)) \neq 1$, $Q(i) \nleqslant G_1(x)$. Since $Q(i) \trianglelefteq G(x_1, x)$, $Q(i)^{\Gamma(x)} \geqslant O_p(G(x_i, x)^{\Gamma(x)})$ by the lemma. It follows that $\langle Q(1), \ldots, Q(n)\rangle$ acts transitively on $\Gamma(x)$. Since $J \leqslant O_p(G(x_1)) \cap \cdots \cap O_p(G(x_n))$, $J \leqslant G_2(x) = 1$. It follows that $|G_1(x_1, x)| \mid q^{(n-1)^2}$.

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