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COHOMOLOGICAL TRIVIALITY BY SPECTRAL METHODS

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ABSTRACT. In this note, the spectral sequence of a group extension is used to obtain a direct proof of the twins' criterion for cohomological triviality of modules over a finite group, stated in its strong form.

The Tate cohomology of finite groups is denoted by $\hat{H}$, while H is reserved for the ordinary group cohomology.

The criterion to be proved is the following.

THEOREM (NAKAYAMA-TATE). *Let G be a finite group and A a G-module. If, for each prime p, there exists an integer r_p (depending on p) such that*

$$\hat{H}^{r_p}(S_p, A) = \hat{H}^{r_p+1}(S_p, A) = 0,$$

where S_p is a Sylow p-subgroup of G, then A is cohomologically trivial.

PROOF. As usual, by the Sylow subgroup argument in cohomology [**3**, Corollary to Theorem 4, p. 148], it suffices to consider the case where G is a p-group, then showing that the G-module A is cohomologically trivial if $\hat{H}^r(G, A) = \hat{H}^{r+1}(G, A) = 0$ for some integer r. This is an immediate consequence of the following two statements:

(i) If $\hat{H}^r(G, A) = \hat{H}^{r+1}(G, A) = 0$, then $\hat{H}^r(S, A) = \hat{H}^{r+1}(S, A) = 0$ for all subgroups S of G.

(ii) If $\hat{H}^r(G, A) = \hat{H}^{r+1}(G, A) = 0$, then $\hat{H}^n(G, A) = 0$ for all integers n.

Since every proper subgroup of G is contained in a normal subgroup of index p in G, to establish (i) S can be taken as such a subgroup, arguing by induction on the order of G. Also, by the standard technique of dimension shifting [**3**, §1, p. 137], it can be assumed that $r = 1$.

Now, let (E, H) be the Hochschild-Serre spectral sequence associated with the G-module A and the subgroup S of G [**2**], so that

$$E_2^{p,q} = H^p(G/S, H^q(S, A)) \quad \text{and} \quad H^n = H^n(G, A).$$

Since $H^1 = 0$, $H^1(S, A)_{G/S} \simeq E_2^{1,1}$, G/S being a cyclic group. For applying the formula $\operatorname{res}_{G,S} \operatorname{cor}_{S,G} = N_{G/S}$ [**1**, Corollary 9.2, p. 257] for dimension 1, $N_{G/S} H^1(S, A) = 0$, and so, $H^1(S, A)_{G/S} \simeq \hat{H}^{-1}(G/S, H^1(S, A))$; but $\hat{H}^{-1}(G/S, H^1(S, A)) \simeq E_2^{1,1}$, by the periodicity of the cohomology of finite cyclic groups [**3**, Corollary to Proposition 6, p. 141]. From $H^1 = 0$ it also

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follows that $E_2^{1,1} \simeq E_3^{1,1}$, because $E_2^{1,0} = 0$ and hence, $E_2^{3,0} = 0$ (periodicity again). Moreover, $E_3^{1,1} = 0$, since $H^2 = 0$. Therefore, it has been proved that $H^1(S, A)_{G/S} = 0$. Now, since $H^1(S, A)$ is annihilated by a power of p [**3**, Corollary 1 to Proposition 4, p. 138], a property of finite p-groups yields $H^1(S, A) = 0$. (This property is well known, at least for modules annihilated by p [**3**, Lemma 4, p. 149]; the case of a module M annihilated by a p-power reduces to the elementary case, by considering M/pM.) Finally, since $H^1(S, A) = 0$, the spectral sequence provides an exact sequence $H^2 \to E_2^{0,2} \to E_2^{3,0}$, where the extreme terms vanish. Thus, $E_2^{0,2} = 0$, which implies $H^2(S, A) = 0$. (This follows from the fact that, if a p-primary module M over a finite p-group K satisfies $M^K = 0$, then $M = 0$. Since $M = \bigcup N$, where N runs through all finitely generated subgroups of M, $M^K = \bigcup N^K$, and the result can be deduced from the finite case [**3**, Lemma 2, p. 146].)

By dimension shifting and by moving up or down one dimension at a time, to prove (ii) it is sufficient to show that:

(iii) If $H^1(G, A) = H^2(G, A) = 0$, then $H^3(G, A) = 0$.

(iv) If $H^2(G, A) = H^3(G, A) = 0$, then $H^1(G, A) = 0$.

Proceeding by induction on the order of G, given a normal subgroup S of index p in G, in the case of (iii) it follows that $H^3(S, A) = 0$, because $H^1(S, A) = H^2(S, A) = 0$, by (i). Then, $E_2^{3,0} \simeq H^3$ (at this point, the spectral sequence is not essential [**3**, Corollary to Proposition 5 (Proof), p. 126]); but $E_2^{3,0} = 0$, as before, since $H^1 = 0$. Similarly, under the hypothesis of (iv), assertion (i) gives $H^2(S, A) = H^3(S, A) = 0$, and hence, by the inductive assumption, $H^1(S, A) = 0$. Therefore, $E_2^{3,0} \simeq H^3 = 0$, so that $H^1 \simeq E_2^{1,0} = 0$. Thus, the proof of the theorem is complete.

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