A BOUND FOR DECOMPOSITIONS OF m-CONVEX SETS WHOSE LNC POINTS LIE IN A HYPERPLANE

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Abstract. A set S in Rd is said to be m-convex, m > 2, if and only if for every m points in S, at least one of the line segments determined by these points lies in S. Let S denote a closed m-convex set in Rd, and assume that the set of lnc points of S lies in a hyperplane. Then S is a union of f(m) or fewer convex sets, where f is defined inductively as follows: f(2) = 1, f(3) = 2, and f(m) = f(m - 2) + 3 for m > 4. Moreover, for d > 3, an example reveals that the best bound is no lower than g(m), where g(m) = f(m) for 2 < m < 5 and for m = 7, and g(m) = g(m - 3) + 4 otherwise.

1. Introduction. Let S be a subset of Rd. The set S is said to be m-convex, m > 2, if and only if for every m distinct points in S, at least one of the line segments determined by these points lies in S. A point x in S is said to be a point of local convexity of S if and only if there is some neighborhood N of x such that N ∩ S is convex. If S fails to be locally convex at some point q in S, then q is called a point of local nonconvexity (lnc point) of S. As in [1], the following terminology will be used: For x, y in S, we say x sees y via S if and only if the corresponding segment [x, y] lies in S. Points x_1, . . . , x_n in S are visually independent via S if and only if for 1 ≤ i < j ≤ n, x_i does not see x_j via S. For x in S, the set of all points of S which see x via S is called the star of x in S, denoted S_x. Throughout the paper, cl S, int S, and rel int S will be used to denote the closure, interior, and relative interior, respectively, of the set S.

Several decomposition theorems have been obtained for closed m-convex sets S in the plane [4], [3], [2], and if we further require the set of lnc points of S to lie on a line, then S will be a union of m - 1 or fewer convex sets [1]. Here we investigate an analogous problem in Rd, d > 3, obtaining a decomposition theorem for closed m-convex sets in Rd whose lnc points lie in a hyperplane. Moreover, an example reveals that for d > 3 and m > 4, the planar bound m - 1 is no longer possible.

2. The results.

Theorem 1. Let S = cl(int S) be an m-convex set in Rd, m > 2, and assume that the set Q of lnc points of S lies in a hyperplane H. Then S is a union of f(m) or fewer convex sets, where the function f is defined inductively as follows:

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$f(2) = 1, f(3) = 2, \text{ and } f(m) = f(m - 2) + 3 \text{ for } m \geq 4.$

Proof. We use an inductive argument. The result is trivial for $m = 2$, and if $m = 3$, it follows immediately from [1, Theorem 1]. Hence suppose that $m \geq 4$ and assume that the theorem is true for all integers $k, 2 \leq k < m$, to prove for $m$.

Let $H_1$ and $H_2$ denote distinct open halfspaces determined by the hyperplane $H$. By the $m$-convexity of $S$, it is easy to see that for $i = 1, 2$, each set $S \cap H_i$ has at most $m - 1$ components, each starshaped. Furthermore, standard arguments involving a result by Valentine [5, Corollary 1] reveal that each of these components is convex. Since $S = \text{cl}(\text{int } S)$, every point of $S$ must lie in the closure of at least one of these components.

We show that, without loss of generality, we may assume every point of $S \cap H$ lies in $\text{cl}(S \cap H_1) \cap \text{cl}(S \cap H_2)$: Otherwise, for some point $x$ in $S \cap H$ and for an appropriate labeling of the halfspaces $H_1$ and $H_2$, $x \notin \text{cl}(S \cap H_1)$. Then since $S = \text{cl}(\text{int } S), x \in \text{cl}(S \cap H_2)$. By previous comments, for some $k, 1 \leq k \leq m - 1$, $x$ is in the closure of exactly $k$ components of $S \cap H_2$, and if we choose one point from each of these components, clearly these points will be visually independent via $S$. If $k = m - 1$, then it is easy to show that $S_x = S, S = \text{cl}(S \cap H_2), \text{ and } S$ is a union of $m - 1 < f(m)$ or fewer convex sets. If $k < m - 2$, then a proof similar to [1, Lemma 1] may be used to show that the set $S' = \text{cl}(S \sim S_x)$ is at most $(m - k)$-convex, $S' = \text{cl}(\text{int } S')$, and the lnc points of $S'$ are in $Q$. Hence by our induction hypothesis, $S'$ is a union of $f(m - k)$ or fewer convex sets. It is not hard to show that $\text{cl}(\text{int } S_x)$ is a union of $k$ convex sets, and we conclude that

$$S = \text{cl}(\text{int } S_x) \cup \text{cl}(S \sim S_x)$$

is a union of $f(m - k) + k < f(m)$ or fewer convex sets, the desired result. Hence throughout the remainder of the proof we may assume that

$$S \cap H \subseteq \text{cl}(S \cap H_1) \cap \text{cl}(S \cap H_2).$$

Similarly, we may assume that for every component $C$ of $S \sim H, \text{cl } C \cap H$ is a full $(d - 1)$-dimensional: Otherwise, for some component $C$ of $S \sim H, S$ will be the union of the convex set $C$, and the $(m - 1)$-convex set $\text{cl}(S \sim C)$, giving a decomposition of $S$ into $1 + f(m - 1) < f(m)$ of fewer convex sets, again the appropriate bound.

To complete the proof, we consider two cases:

Case 1. Assume for the moment that there exist some lnc point $q$ of $S$ and some component $C$ of $S \sim H$ such that $q \in \text{rel int(cl } C \cap H).$ (Recall that $\text{cl } C \cap H$ is $(d - 1)$-dimensional.) For convenience of notation, let $C \subseteq H_1$. Then $q$ is in the closure of $k$ components of $S \cap H_2$. Moreover, since we are assuming that

$$S \cap H \subseteq \text{cl}(S \cap H_1) \cap \text{cl}(S \cap H_2),$$

it is clear that $2 < k$. Hence $2 < k < m - 1$. It is easy to see that the set
cl(int S_q) is a union of k + 1 convex sets, each the closure of a component of S ~ H. If k = m - 1, then standard arguments show that S = S_q, and S is a union of k + 1 = m < f(m) or fewer convex sets. If k < m - 2, then again the proof of [1, Lemma 1] reveals that the set cl(S ~ S_q) is at most (m - k)-convex and satisfies our induction hypothesis. Thus cl(S ~ S_q) is a union of f(m - k) or fewer convex sets, and S is a union of f(m - k) + k + 1 convex sets. But for 2 < k < m - 2,

\[ f(m - k) + k - 2 < f(m - k + k - 2) = f(m - 2), \]

so \( f(m - k) + k + 1 < f(m - 2) + 3 = f(m) \), and S is a union of f(m) or fewer convex sets, finishing the argument for Case 1.

Case 2. If Case 1 does not occur, then select any component C of S \( \cap H_1 \). By previous assumptions, there exists a component D of S \( \cap H_2 \) such that

\[ \text{rel int}(\text{cl } C \cap H) \cap \text{rel int}(\text{cl } D \cap H) \neq \emptyset. \]

Moreover, D must be unique: Otherwise, for appropriate components D_1 and D_2 of S \( \cap H_2 \), cl D_1 \( \cap \) cl D_2 \( \cap \) rel int(cl C \( \cap \) H) \( \neq \) \( \emptyset \), and there would be some inc point of S in rel int(cl C \( \cap \) H), contradicting our hypothesis for Case 2. Similarly, C is the unique component of S \( \cap H_1 \) for which rel int(cl C \( \cap \) H) \( \cap \) rel int(cl D \( \cap \) H) \( \neq \) \( \emptyset \), and cl C \( \cap \) H = cl D \( \cap \) H. Note that the set cl C \( \cup \) cl D is j-convex, where j is either 2 or 3. Now since S = cl(int S) and cl C \( \cap \) H = cl D \( \cap \) H, it is easy to show that for any point z in cl C \( \cup \) cl D and for any w in S \( \sim \) (cl C \( \cup \) cl D), there is a sequence \( \{w_n\} \) in S converging to w such that no \( w_n \) sees z via S. Hence standard arguments similar to those in [1, Lemma 1] may be used to show that the set cl(S \( \sim \) (cl C \( \cup \) cl D)) is \( (m - j + 1) \)-convex and satisfies our induction hypothesis. We conclude that S is a union of \( (j - 1) + f(m - j + 1) < f(m) \) or fewer convex sets. This completes Case 2 and finishes the proof of the theorem.

It is interesting to note that for \( d > 3 \), without the requirement that S = cl(int S), the theorem fails and, in fact, no finite decomposition is possible. (See [1, Remarks 1 and 2].)

In conclusion, the following example reveals that for \( d > 3 \), the best bound is no lower than \( g(m) \), where \( g(m) = f(m) \) for \( 2 < m < 5 \) and for \( m = 7 \), and \( g(m) = g(m - 3) + 4 \) otherwise.

Example 1. Let g be a function defined inductively as follows: \( g(2) = 1 \), \( g(3) = 2 \), \( g(4) = 4 \), and \( g(m) = g(m - 3) + 4 \) for \( m > 5 \). Then for each \( m > 2 \) and \( d > 3 \), there corresponds an m-convex set \( T_{md} \) in \( R_d \) which satisfies the hypothesis of Theorem 1 and which is expressible as a union of \( g(m) \) and no fewer convex sets.

Examples for \( m = 2 \) and \( m = 3 \) are trivial. Hence let \( m = 4 \) and, for convenience of notation, let \( d = 3 \). Define \( x_1 = (2, 0, 0) \), \( x_2 = (0, 2, 0) \), \( x_3 = (-2, 0, 0) \), \( x_4 = (0, -2, 0) \). Define \( R \equiv \text{conv}(x_j: j \neq i) \), \( 1 < i < 4 \), and let \( y_1 = (-1, 0, 1) \), \( y_3 = (1, 0, 1) \), \( y_2 = (0, -1, -1) \), \( y_4 = (0, 1, -1) \). (Note that,
looking down on the $x$-$y$ plane from the positive $z$ axis, $y_i$ is above the corresponding $R_i$ set for $i = 1, 3$, below for $i = 2, 4$.) Finally, let

$$T_{4,3} = \bigcup \{ \text{conv}(R_i \cup \{y_i\}): 1 < i < 4 \}.$$

Then $T_{4,3}$ is a 4-convex set which is a union of 4 and no fewer convex sets. This example may be generalized to higher dimensions to construct $T_{4,d}$.

Inductively, for $m > 5$, let $T_{m,d}$ be the disjoint union of $T_{m-3,d}$ and $T_{4,d}$. Then $T_{m,d}$ will be an $m$-convex set in $R^d$ expressible as a union of $g(m-3) + 4 = g(m)$ and no fewer convex sets. Hence the planar bound of $m - 1$ fails when $d > 3$ and $m > 4$.

REFERENCES


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