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A BOUND FOR DECOMPOSITIONS OF m-CONVEX SETS WHOSE LNC POINTS LIE IN A HYPERPLANE[1]

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ABSTRACT. A set S in R^d is said to be m-convex, $m \geqslant 2$, if and only if for every m points in S, at least one of the line segments determined by these points lies in S. Let S denote a closed m-convex set in R^d, and assume that the set of lnc points of S lies in a hyperplane. Then S is a union of $f(m)$ or fewer convex sets, where f is defined inductively as follows: $f(2) = 1$, $f(3) = 2$, and $f(m) = f(m-2) + 3$ for $m \geqslant 4$. Moreover, for $d \geqslant 3$, an example reveals that the best bound is no lower than $g(m)$, where $g(m) = f(m)$ for $2 \leqslant m \leqslant 5$ and for $m = 7$, and $g(m) = g(m-3) + 4$ otherwise.

1. Introduction. Let S be a subset of R^d. The set S is said to be *m-convex*, $m \geqslant 2$, if and only if for every m distinct points in S, at least one of the line segments determined by these points lies in S. A point x in S is said to be a *point of local convexity* of S if and only if there is some neighborhood N of x such that $N \cap S$ is convex. If S fails to be locally convex at some point q in S, then q is called a *point of local nonconvexity* (lnc point) of S. As in **[1]**, the following terminology will be used: For x, y in S, we say x *sees* y *via* S if and only if the corresponding segment $[x, y]$ lies in S. Points $x_1, \ldots, x_n$ in S are *visually independent via* S if and only if for $1 \leqslant i < j \leqslant n$, x_i does not see x_j via S. For x in S, the set of all points of S which see x via S is called the *star of* x *in* S, denoted S_x. Throughout the paper, cl S, int S, and rel int S will be used to denote the closure, interior, and relative interior, respectively, of the set S.

Several decomposition theorems have been obtained for closed m-convex sets S in the plane **[4]**, **[3]**, **[2]**, and if we further require the set of lnc points of S to lie on a line, then S will be a union of $m - 1$ or fewer convex sets **[1]**. Here we investigate an analogous problem in R^d, $d \geqslant 3$, obtaining a decomposition theorem for closed m-convex sets in R^d whose lnc points lie in a hyperplane. Moreover, an example reveals that for $d \geqslant 3$ and $m \geqslant 4$, the planar bound $m - 1$ is no longer possible.

2. The results.

THEOREM 1. *Let $S = \text{cl}(\text{int } S)$ be an m-convex set in R^d, $m \geqslant 2$, and assume that the set Q of lnc points of S lies in a hyperplane H. Then S is a union of $f(m)$ or fewer convex sets, where the function f is defined inductively as follows:*

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$f(2) = 1, f(3) = 2$, *and* $f(m) = f(m-2) + 3$ *for* $m \geqslant 4$.

PROOF. We use an inductive argument. The result is trivial for $m = 2$, and if $m = 3$, it follows immediately from [**1**, Theorem 1]. Hence suppose that $m \geqslant 4$ and assume that the theorem is true for all integers k, $2 \leqslant k < m$, to prove for m.

Let H_1 and H_2 denote distinct open halfspaces determined by the hyperplane H. By the m-convexity of S, it is easy to see that for $i = 1, 2$, each set $S \cap H_i$ has at most $m - 1$ components, each starshaped. Furthermore, standard arguments involving a result by Valentine [**5**, Corollary 1] reveal that each of these components is convex. Since $S = \mathrm{cl}(\mathrm{int}\, S)$, every point of S must lie in the closure of at least one of these components.

We show that, without loss of generality, we may assume every point of $S \cap H$ lies in $\mathrm{cl}(S \cap H_1) \cap \mathrm{cl}(S \cap H_2)$: Otherwise, for some point x in $S \cap H$ and for an appropriate labeling of the halfspaces H_1 and H_2, $x \notin \mathrm{cl}(S \cap H_1)$. Then since $S = \mathrm{cl}(\mathrm{int}\, S)$, $x \in \mathrm{cl}(S \cap H_2)$. By previous comments, for some k, $1 \leqslant k \leqslant m - 1$, x is in the closure of exactly k components of $S \cap H_2$, and if we choose one point from each of these components, clearly these points will be visually independent via S. If $k = m - 1$, then it is easy to show that $S_x = S$, $S = \mathrm{cl}(S \cap H_2)$, and S is a union of $m - 1 < f(m)$ or fewer convex sets. If $k \leqslant m - 2$, then a proof similar to [**1**, Lemma 1] may be used to show that the set $S' \equiv \mathrm{cl}(S \sim S_x)$ is at most $(m-k)$-convex, $S' = \mathrm{cl}(\mathrm{int}\, S')$, and the lnc points of S' are in Q. Hence by our induction hypothesis, S' is a union of $f(m-k)$ or fewer convex sets. It is not hard to show that $\mathrm{cl}(\mathrm{int}\, S_x)$ is a union of k convex sets, and we conclude that

$$S = \mathrm{cl}(\mathrm{int}\, S_x) \cup \mathrm{cl}(S \sim S_x)$$

is a union of $f(m-k) + k \leqslant f(m)$ or fewer convex sets, the desired result. Hence throughout the remainder of the proof we may assume that

$$S \cap H \subseteq \mathrm{cl}(S \cap H_1) \cap \mathrm{cl}(S \cap H_2).$$

Similarly, we may assume that for every component C of $S \sim H$, $\mathrm{cl}\, C \cap H$ is a full $(d-1)$-dimensional: Otherwise, for some component C of $S \sim H$, S will be the union of the convex set C, and the $(m-1)$-convex set $\mathrm{cl}(S \sim C)$, giving a decomposition of S into $1 + f(m-1) \leqslant f(m)$ of fewer convex sets, again the appropriate bound.

To complete the proof, we consider two cases:

Case 1. Assume for the moment that there exist some lnc point q of S and some component C of $S \sim H$ such that $q \in \mathrm{rel\ int}(\mathrm{cl}\, C \cap H)$. (Recall that $\mathrm{cl}\, C \cap H$ is $(d-1)$-dimensional.) For convenience of notation, let $C \subseteq H_1$. Then q is in the closure of k components of $S \cap H_2$. Moreover, since we are assuming that

$$S \cap H \subseteq \mathrm{cl}(S \cap H_1) \cap \mathrm{cl}(S \cap H_2),$$

it is clear that $2 \leqslant k$. Hence $2 \leqslant k \leqslant m - 1$. It is easy to see that the set cl(int S_q) is a union of $k + 1$ convex sets, each the closure of a component of $S \sim H$. If $k = m - 1$, then standard arguments show that $S = S_q$, and S is a union of $k + 1 = m \leqslant f(m)$ or fewer convex sets. If $k \leqslant m - 2$, then again the proof of [**1**, Lemma 1] reveals that the set cl($S \sim S_q$) is at most ($m - k$)-convex and satisfies our induction hypothesis. Thus cl($S \sim S_q$) is a union of $f(m - k)$ or fewer convex sets, and S is a union of $f(m - k) + k + 1$ convex sets. But for $2 \leqslant k \leqslant m - 2$,

$$f(m - k) + k - 2 \leqslant f(m - k + k - 2) = f(m - 2),$$

so $f(m - k) + k + 1 \leqslant f(m - 2) + 3 = f(m)$, and S is a union of $f(m)$ or fewer convex sets, finishing the argument for Case 1.

Case 2. If Case 1 does not occur, then select any component C of $S \cap H_1$. By previous assumptions, there exists a component D of $S \cap H_2$ such that

$$\text{rel int}(\text{cl } C \cap H) \cap \text{rel int}(\text{cl } D \cap H) \neq \varnothing.$$

Moreover, D must be unique: Otherwise, for appropriate components D_1 and D_2 of $S \cap H_2$, cl $D_1 \cap$ cl $D_2 \cap$ rel int(cl $C \cap H) \neq \varnothing$, and there would be some lnc point of S in rel int(cl $C \cap H$), contradicting our hypothesis for Case 2. Similarly, C is the unique component of $S \cap H_1$ for which rel int(cl $C \cap H) \cap$ rel int(cl $D \cap H) \neq \varnothing$, and cl $C \cap H =$ cl $D \cap H$. Note that the set cl $C \cup$ cl D is j-convex, where j is either 2 or 3. Now since $S =$ cl(int S) and cl $C \cap H =$ cl $D \cap H$, it is easy to show that for any point z in cl $C \cup$ cl D and for any w in $S \sim$ (cl $C \cup$ cl D), there is a sequence $\{w_n\}$ in S converging to w such that no w_n sees z via S. Hence standard arguments similar to those in [**1**, Lemma 1] may be used to show that the set cl($S \sim$ (cl $C \cup$ cl D)) is $(m - j + 1)$-convex and satisfies our induction hypothesis. We conclude that S is a union of $(j - 1) + f(m - j + 1) \leqslant f(m)$ or fewer convex sets. This completes Case 2 and finishes the proof of the theorem.

It is interesting to note that for $d \geqslant 3$, without the requirement that $S =$ cl(int S), the theorem fails and, in fact, no finite decomposition is possible. (See [**1**, Remarks 1 and 2].)

In conclusion, the following example reveals that for $d \geqslant 3$, the best bound is no lower than $g(m)$, where $g(m) = f(m)$ for $2 \leqslant m \leqslant 5$ and for $m = 7$, and $g(m) = g(m - 3) + 4$ otherwise.

EXAMPLE 1. Let g be a function defined inductively as follows: $g(2) = 1$, $g(3) = 2$, $g(4) = 4$, and $g(m) = g(m - 3) + 4$ for $m \geqslant 5$. Then for each $m \geqslant 2$ and $d \geqslant 3$, there corresponds an m-convex set $T_{m,d}$ in R^d which satisfies the hypothesis of Theorem 1 and which is expressible as a union of $g(m)$ and no fewer convex sets.

Examples for $m = 2$ and $m = 3$ are trivial. Hence let $m = 4$ and, for convenience of notation, let $d = 3$. Define $x_1 = (2, 0, 0)$, $x_2 = (0, 2, 0)$, $x_3 = (-2, 0, 0)$, $x_4 = (0, -2, 0)$. Define $R_i \equiv \text{conv}\{x_j : j \neq i\}$, $1 \leqslant i \leqslant 4$, and let $y_1 = (-1, 0, 1)$, $y_3 = (1, 0, 1)$, $y_2 = (0, -1, -1)$, $y_4 = (0, 1, -1)$. (Note that, looking down on the x-y plane from the positive z axis, y_i is above the corresponding R_i set for $i = 1, 3$, below for $i = 2, 4$.) Finally, let

$$T_{4,3} = \bigcup \{\operatorname{conv}(R_i \cup \{y_i\}) : 1 \leqslant i \leqslant 4\}.$$

Then $T_{4,3}$ is a 4-convex set which is a union of 4 and no fewer convex sets. This example may be generalized to higher dimensions to construct $T_{4,d}$.

Inductively, for $m \geqslant 5$, let $T_{m,d}$ be the disjoint union of $T_{m-3,d}$ and $T_{4,d}$. Then $T_{m,d}$ will be an m-convex set in R^d expressible as a union of $g(m - 3) + 4 = g(m)$ and no fewer convex sets. Hence the planar bound of $m - 1$ fails when $d \geqslant 3$ and $m \geqslant 4$.

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