PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 72, Number 1, October 1978

ANALYTICITY OF FUNCTIONS AND SUBALGEBRAS OF L^∞ CONTAINING H^∞

S.-Y. A. CHANG[1] AND J. B. GARNETT[2]

ABSTRACT. Let B be a subalgebra of L^∞ containing H^∞. We give some necessary and sufficient conditions, expressed in terms of analyticity, for a function in L^∞ to belong to B.

1. Let H^∞ be the algebra of bounded analytic functions on the open disc D. By Fatou's theorem H^∞ is a closed subalgebra of L^∞, the algebra of essentially bounded Lebesgue measurable functions on the unit circle C. The (closed) subalgebras of L^∞ containing H^∞ have received considerable attention recently (cf. D. Sarason [7], [8], S.-Y. Chang [3], [4] and D. Marshall [6]). The main result of those papers is that each such algebra B is a Douglas algebra, i.e. B is generated by H^∞ and

$$\mathfrak{B} = \{\bar{b} : b \in H^\infty \text{ is an inner function and } \bar{b} \in B\}.$$

In this note we characterize the elements of B in terms of their analyticity in two different ways.

We identify $f \in L^1$ with its Poisson integral

$$f(re^{i\theta}) = \frac{1}{2\pi} \int_{-\pi}^{\pi} P_r(\theta - \varphi) f(\varphi)\, d\varphi$$

where $P_r(t) = (1 - r^2)/(1 - 2r\cos t + r^2)$ is the Poisson kernel. For $\delta > 0$ we let $G_\delta(f)$ be the region $\{re^{i\theta}: |f(re^{i\theta})| > 1 - \delta\}$. For each arc I on the circle with center e^{it} and normalized arc length $|I|$, we let $\mathfrak{R}(I)$ be the region

$$\{re^{i\theta}: |\theta - t| \leqslant |I|/2,\ 1 - |I| < r < 1\}.$$

We write $H^\infty(G)$ for the set of bounded analytic functions on a region G.

2. The first characterization connects the algebra B to the algebras $H^\infty(G_\delta(b))$, $0 < \delta < 1$, $b \in \overline{\mathfrak{B}} = \{b \text{ inner}: \bar{b} \in B\}$.

LEMMA 2.1. *Let $b(z)$ be an inner function and let $0 < \delta < 1$. Then almost every $e^{i\theta} \in C$ is the vertex of a truncated cone lying in $G_\delta(b)$. Every bounded harmonic function $F(z)$ defined on $G_\delta(b)$ has a nontangential limit $F(e^{i\theta})$ at almost every $e^{i\theta} \in C$.*

PROOF. At almost every $e^{i\theta}$, $b(z)$ has a unimodular nontangential limit.

Received by the editors September 22, 1977.
AMS (MOS) subject classifications (1970). Primary 30A76, 43A40.
[1]Partially supported by NSF Grant MCS 77-16281.
[2]Partially supported by NSF Grant MPS 74-7035.

Such a point $e^{i\theta}$ is then the vertex of a truncated cone (of arbitrarily large aperture) inside $G_\delta(b)$. For any $\varepsilon > 0$ and any $\alpha > 0$, a metric density argument [9, p. 201] shows there is h, $0 < h < 1$, and there is a compact set $E \subset C$ such that $|C \setminus E| < \varepsilon$ and such that $G_\delta(b)$ contains

$$\mathfrak{R} = \bigcup_{e^{i\theta} \in E} \left\{ z: \frac{|e^{i\theta} - z|}{1 - |z|} < \alpha, 1 - h < |z| < 1 \right\}.$$

Now the proof commencing on the bottom of p. 202 of [9] shows that every bounded harmonic function on $\mathfrak{R}$ has a nontangential limit from within $\mathfrak{R}$ at almost every point of E. Since ε and α are arbitrary the lemma is proved.

Because of the lemma we can state the following

THEOREM 2.2. *Let $f \in L^\infty$. Then $f \in B$ if and only if for every $\varepsilon > 0$ there is $b \in \mathfrak{B}$ and δ, $0 < \delta < 1$, and there exists $F \in H^\infty(G_\delta(b))$ with nontangential limit $F(e^{i\theta})$ such that $\|F(e^{i\theta}) - f(e^{i\theta})\|_\infty < \varepsilon$.*

PROOF. First suppose $f \in B$. Then f can be uniformly approximated by functions of the form $\bar{b}h$ with $b \in \mathfrak{B}$ and $h \in H^\infty$. When $0 < \delta < 1$, $F(z) = h(z)/b(z)$ is in $H^\infty(G_\delta(b))$ and F has nontangential limit $h(e^{i\theta})/b(e^{i\theta}) = \bar{b}(e^{i\theta})h(e^{i\theta})$ almost everywhere. Thus the condition of the theorem is necessary.

The proof of the converse uses the basic construction from the proof of the corona theorem.

LEMMA 2.3. *Let $b(z)$ be an inner function and let $0 < \eta < 1$. There is a sequence Γ_i of disjoint rectifiable Jordan curves bounding domains $D_i \subset D$ such that:*

$$\{|b(z)| < \eta\} \subset \bigcup D_i. \tag{2.1}$$

$$\inf_{D_i} |b(z)| < \eta. \tag{2.2}$$

(2.3) *There is $\delta = \delta(\eta) < 1$ such that $\Gamma_i \subset \{|b(z)| < \delta(\eta)\}$.*

(2.4) *Arc length in $\Gamma = D \cap \bigcup_i \Gamma_i$ is a Carleson measure on D.*

See [1], [2] or [10] for detailed proofs of Lemma 2.3.

To conclude the proof of Theorem 1, it suffices to assume that $f = F$ almost everywhere, where $F \in H^\infty(G_\delta(b))$ for some $b \in \overline{\mathfrak{B}}$ and some δ, $0 < \delta < 1$. Using the duality $L^\infty/H^\infty = (H_0^1)^*$, we have for $n = 1, 2, \ldots$

$$\begin{aligned} \operatorname{dist}(f, B) &\leqslant \inf_{h \in H^\infty} \|f - \bar{b}^n h\| \\ &= \sup_{\substack{g \in H^1 \\ \|g\|_1 < 1}} \left| \frac{1}{2\pi i} \int_C F(z) b^n(z) g(z)\, dz \right|. \end{aligned}$$

Take $\eta > 1 - \delta$ and consider the curves Γ_i given by Lemma 2.2. Let

$$\Omega_r = \{|z| < r\} \setminus \bigcup_i \bar{D}_i, \qquad r < 1.$$

By (2.2) the region Ω_r has finite connectivity, and since $b(z)$ is inner, $\overline{\Omega}_r \subset G_\delta(b)$ by (2.1) when $\eta > 1 - \delta$. Moreover Ω_r has rectifiable boundary consisting of

$$J_r = \{|z| = r\} \cap \partial\Omega_r$$

and

$$K_r = \{|z| < r\} \cap \bigcup_i \Gamma_i.$$

By (2.3) for almost every $e^{i\theta}$, $re^{i\theta} \in J_r$ when $1 - r$ is small. Hence by dominated convergence and Lemma 2.1

$$\lim_{r \to 1} \int_{J_r} F(z) b^n(z) g(z)\, dz = \int_C F(z) b^n(z) g(z)\, dz.$$

By Cauchy's theorem

$$\int_{J_r} F(z) b^n(z) g(z)\, dz = -\int_{K_r} F(z) b^n(z) g(z)\, dz$$

with correct orientations. But by (2.3) and (2.4)

$$\int_{K_r} |F(z)|\, |b^n(z)|\, |g(z)|\, ds \leqslant \sup|F(z)| (\delta(\eta))^n M \|g\|_1,$$

where M depends only on Γ. Sending $n \to \infty$ completes the proof.

The theorem or its proof shows that $\operatorname{dist}(f, B)$ is the infimum of those $\varepsilon > 0$ for which the condition of the theorem remains true.

3. The second characterization of B involves the distances from f to H^2, measured in the Hilbert spaces $L^2(P_{r_0}(\theta - \theta_0)d\theta)$ for points $z_0 = r_0 e^{i\theta_0}$ lying in some region $G_\delta(b)$, $b \in \overline{B}$.

For $f \in L^\infty$, let

$$d\mu_f = |\partial f / \partial \bar{z}|^2 (1 - |z|)\, dx\, dy$$

where $\partial/\partial\bar{z} = \frac{1}{2}(\partial/\partial x + \partial i/\partial y)$. The Littlewood-Paley identity

$$\frac{1}{\pi} \int\int |\nabla f(z)|^2 \log \frac{1}{|z|}\, dx\, dy = \frac{1}{2\pi} \int |f - f(0)|^2\, d\theta,$$

where $|\nabla f|^2 = |\partial f/\partial x|^2 + |\partial f / \partial y|^2$, implies that $d\mu_f$ is a finite measure on D.

THEOREM 3.1. *When $f \in L^\infty$ the following conditions are equivalent.*

(i) $f \in B$.

(ii) *For any $\varepsilon > 0$ there is $b \in \overline{\mathfrak{B}}$ and δ, $0 < \delta < 1$, such that for all $z_0 \in G_\delta(b)$,*

$$\inf_{g \in H^2} \frac{1}{2\pi} \int |f - g|^2 P_{r_0}(\theta - \theta_0)\, d\theta < \varepsilon. \tag{3.1}$$

(iii) *For any $\varepsilon > 0$ there is $b \in \overline{\mathfrak{B}}$ and δ, $0 < \delta < 1$, such that*

$$\sup_I \frac{\mu_f(G_\delta(b) \cap \mathfrak{R}(I))}{|I|} < \varepsilon. \tag{3.2}$$

PROOF. We show (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (i).

Assume (i) holds. Then there is $\bar{b} \in \mathfrak{B}$ and $h \in H^\infty$ such that $\|f - \bar{b}h\|_\infty < \varepsilon$. For $z_0 \in G_\delta(b)$, let $g(z) = \overline{b(z_0)}h(z)$. Then $g \in H^2$ and

$$\frac{1}{2\pi} \int |\bar{b}h - g|^2 P_{r_0}(\theta - \theta_0)\, d\theta \leqslant \frac{\|h\|_\infty}{2\pi} \int |b(\theta) - b(z_0)|^2 P_{r_0}(\theta - \theta_0)\, d\theta$$
$$= \|h\|_\infty \left(1 - |b(z_0)|^2\right) \leqslant 2\delta \|h\|_\infty.$$

Consequently, (3.1) holds if δ is sufficiently small.

Now suppose (ii) holds and choose $\bar{b} \in \mathfrak{B}$ and δ so that (3.1) holds. We follow the proof of Lemma 2 of [3]. By Lemma 5 of [3], (3.2) will be proved if we show that

$$\mu_f(\mathfrak{R}(I_0)) < \varepsilon |I_0|$$

for all arcs I_0 of the form $\{|\theta - \theta_0| < 1 - r_0\}$ where $z_0 = r_0 e^{i\theta_0} \in G_\delta(b)$. Let $w = (z - z_0)/(1 - \bar{z}_0 z)$ and let $F(w) = f(z) - g(z)$, where $g \in H^2$ is chosen to attain the infimum (3.1). Then $F(w)$ is conjugate analytic, so that $|\nabla F(w)|^2 = 2|\partial F/\partial \bar{w}|^2$. Since $F(0) = 0$, the Littlewood-Paley identity gives us

$$\frac{1}{2\pi} \int_0^{2\pi} |F(e^{i\varphi})|^2\, d\varphi = \frac{2}{\pi} \int\int \left| \frac{\partial F}{\partial \bar{w}} \right|^2 \log \frac{1}{|w|}\, du\, dv$$

where $w = u + iv$. A change of variables then yields

$$\frac{1}{2\pi} \int_0^{2\pi} |f(\theta) - g(\theta)|^2 P_{r_0}(\theta - \theta_0)\, d\theta$$
$$= \frac{2}{\pi} \int\int \left| \frac{\partial f}{\partial \bar{z}} \right|^2 \log \left| \frac{1 - \bar{z}_0 z}{z - z_0} \right| dx\, dy,$$

as $\partial g/\partial \bar{z} = 0$. When $z \in \mathfrak{R}(I_0)$,

$$\frac{1 - |z|}{1 - r_0} \leqslant c \log \left| \frac{1 - \bar{z}_0 z}{z - z_0} \right|,$$

and hence (3.1) implies that

$$\mu_f(\mathfrak{R}(I_0)) \leqslant \frac{c(1 - r_0)}{2\pi} \int |f - g|^2 P_{r_0}(\theta - \theta_0)\, d\theta < c\varepsilon(1 - r_0).$$

Now assume (iii). Let $\varepsilon > 0$ and fix $b \in \overline{\mathfrak{B}}$ and δ, $0 < \delta < 1$, so that (3.2) holds. We estimate

$$\operatorname{dist}(f, \bar{b}^n H^\infty) = \sup_{g \in H_0^1} \frac{1}{2\pi} \int f b^n g\, d\theta$$

as in the proof of Theorem 6 of [3] with one small modification. Note that when $g \in H^1$,

$$\nabla f(z) \nabla (b^n g)(z) = f_x (b^n g)_x + f_y (b^n g)_y = 2(\partial f/\partial \bar{z})(\partial (b^n g)/\partial z).$$

Polarization of the Littlewood-Paley identity then yields

$$\frac{1}{2\pi}\int fb^n g\, d\theta = \frac{2}{\pi}\iint \frac{\partial f}{\partial \bar z}\frac{\partial b^n g}{\partial z}\log\frac{1}{|z|}\,dx\,dy.$$

From this point one can repeat the proof of Theorem 6 in [3], using (3.2) instead of the analogous condition on $|\nabla f|^2(1-|z|)\,dx\,dy$, and obtain

$$\operatorname{dist}\left(f, \bar b^n H^\infty\right) \leqslant C\varepsilon^{1/2}.$$

Thus $f \in B$ if (iii) holds and the theorem is proved.

The proof of the theorem contains the following estimates on dist(f, B) for $f \in L^\infty$. Let $\varepsilon_1(f)$ be the infimum of those $\varepsilon > 0$ for which condition (ii) is true and let $\varepsilon_2(f)$ be the infimum of those $\varepsilon > 0$ for which condition (iii) is true. Then

$$\operatorname{dist}(f, B) \geqslant c_1\varepsilon_1^{1/2} \geqslant c_2\varepsilon_2^{1/2} \geqslant c_3 \operatorname{dist}(f, B),$$

for universal constants c_1, c_2 and c_3. (These inequalities, reading from the left, follow from the proofs of (i) $\Rightarrow$ (ii), (ii) $\Rightarrow$ (iii), (iii) $\Rightarrow$ (i) respectively.)

Since $\partial \bar f/\partial \bar z = \overline{(\partial f/\partial z)}$, the description of $B \cap \bar B$ given as Theorem 8 in [3] is an immediate corollary of Theorem 3.1.

Corollary 3.2. *If $f \in L^\infty$, then $f \in B$ if and only if for any $\varepsilon > 0$ there is $b \in \bar{\mathfrak{B}}$ and δ, $0 < \delta < 1$, such that*

$$\iint_{\mathfrak{R}(I)\cap G_\delta(b)} |\nabla f|^2(1-|z|)\,dx\,dy < \varepsilon(I)$$

for every arc I.

References

1. L. Carleson, *Interpolations by bounded analytic functions and the corona problem*, Ann. of Math. (2) **76** (1962), 547–559.

2. ______, *The corona theorem*, Proc. of the 15th Scandinavian Congress (Oslo, 1968), Lecture Notes in Math., vol. 118, Springer-Verlag, Berlin and New York, 1970.

3. S.-Y. Chang, *A characterization of Douglas subalgebras*, Acta Math. **137** (1976), 81–90.

4. ______, *Structure of subalgebras between L^∞ and H^∞*, Trans. Amer. Math. Soc. **227** (1977), 319–332.

5. C. Fefferman and E. M. Stein, *H^p spaces of several variables*, Acta Math. **129** (1972), 137–193.

6. D. E. Marshall, *Subalgebras of L^∞ containing H^∞*, Acta Math. **137** (1976), 91–98.

7. D. Sarason, *Algebras of functions on the unit circle*, Bull. Amer. Math. Soc. **79** (1973), 286–299.

8. ______, *Algebras between L^∞ and H^∞*, Spaces of Analytic Functions (Kristiansen, Norway, 1975), Lecture Notes in Math., vol. 512, Springer-Verlag, Berlin and New York.

9. E. M. Stein, *Singular integrals and differentiability properties of functions*, Princeton Univ. Press, Princeton, N. J., 1970.

10. S. Ziskind, *Interpolating sequences and the Silov boundary of $H^\infty(\Delta)$*, J. Functional Analysis **21** (1976), 380–388.

Department of Mathematics, University of Maryland, College Park, Maryland 20742

Department of Mathematics, University of California, Los Angeles, California 90024