ANALYTICITY OF FUNCTIONS AND SUBALGEBRAS OF $L^\infty$ CONTAINING $H^\infty$

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Abstract. Let $B$ be a subalgebra of $L^\infty$ containing $H^\infty$. We give some necessary and sufficient conditions, expressed in terms of analyticity, for a function in $L^\infty$ to belong to $B$.

1. Let $H^\infty$ be the algebra of bounded analytic functions on the open disc $D$. By Fatou's theorem $H^\infty$ is a closed subalgebra of $L^\infty$, the algebra of essentially bounded Lebesgue measurable functions on the unit circle $C$. The (closed) subalgebras of $L^\infty$ containing $H^\infty$ have received considerable attention recently (cf. D. Sarason [7], [8], S.-Y. Chang [3], [4] and D. Marshall [6]). The main result of those papers is that each such algebra $B$ is a Douglas algebra, i.e. $B$ is generated by $H^\infty$ and

$$ \mathfrak{B} = \{ \tilde{b} : b \in H^\infty \text{ is an inner function and } \tilde{b} \in B \}.$$ 

In this note we characterize the elements of $B$ in terms of their analyticity in two different ways. We identify $f \in L^1$ with its Poisson integral

$$ f(re^{i\theta}) = \frac{1}{2\pi} \int_{-\pi}^{\pi} P_r(\theta - \varphi) f(\varphi) \, d\varphi $$

where $P_r(t) = (1 - r^2)/(1 - 2r \cos t + r^2)$ is the Poisson kernel. For $\delta > 0$ we let $G_\delta(f)$ be the region $\{re^{i\theta} : |f(re^{i\theta})| > 1 - \delta\}$. For each arc $I$ on the circle with center $e_\omega$ and normalized arc length $|I|$, we let $\mathcal{G}(I)$ be the region $\{re^{i\theta} : |\theta - t| < |I|/2, 1 - |I| < r < 1\}$.

We write $H^\infty(G)$ for the set of bounded analytic functions on a region $G$.

2. The first characterization connects the algebra $B$ to the algebras $H^\infty(G_\delta(b))$, $0 < \delta < 1$, $b \in \mathfrak{B} = \{ \text{inner: } \tilde{b} \in B \}$.

Lemma 2.1. Let $b(z)$ be an inner function and let $0 < \delta < 1$. Then almost every $e^{i\theta} \in C$ is the vertex of a truncated cone lying in $G_\delta(b)$. Every bounded harmonic function $F(z)$ defined on $G_\delta(b)$ has a nontangential limit $F(e^{i\theta})$ at almost every $e^{i\theta} \in C$.

Proof. At almost every $e^{i\theta}$, $b(z)$ has a unimodular nontangential limit.
Such a point \( e^{i\theta} \) is then the vertex of a truncated cone (of arbitrarily large aperture) inside \( G_b(b) \). For any \( \varepsilon > 0 \) and any \( \alpha > 0 \), a metric density argument \([9, \text{p. 201}]\) shows there is \( h, 0 < h < 1 \), and there is a compact set \( E \subset C \) such that \( |C \setminus E| < \varepsilon \) and such that \( G_b(b) \) contains

\[
\mathcal{R} = \bigcup_{e^{i\theta} \in E} \left\{ z : \frac{|e^{i\theta} - z|}{1 - |z|} < \alpha, 1 - h < |z| < 1 \right\}.
\]

Now the proof commencing on the bottom of p. 202 of [9] shows that every bounded harmonic function on \( \mathcal{R} \) has a non-tangential limit from within \( \mathcal{R} \) at almost every point of \( E \). Since \( \varepsilon \) and \( \alpha \) are arbitrary the lemma is proved.

Because of the lemma we can state the following

**Theorem 2.2.** Let \( f \in L^\infty \). Then \( f \in B \) if and only if for every \( \varepsilon > 0 \) there is \( b \in \partial \mathbb{D} \) and \( \delta, 0 < \delta < 1 \), and there exists \( F \in H^\infty(G_b(b)) \) with non-tangential limit \( F(e^{i\theta}) \) such that \( \|F(e^{i\theta}) - f(e^{i\theta})\|_\infty < \varepsilon \).

**Proof.** First suppose \( f \in B \). Then \( f \) can be uniformly approximated by functions of the form \( \bar{b}h \) with \( b \in \partial \mathbb{D} \) and \( h \in H^\infty \). When \( 0 < \delta < 1 \), \( F(z) = h(z)/b(z) \) is in \( H^\infty(G_b(b)) \) and \( F \) has non-tangential limit \( h(e^{i\theta})/b(e^{i\theta}) = \bar{b}(e^{i\theta})h(e^{i\theta}) \) almost everywhere. Thus the condition of the theorem is necessary.

The proof of the converse uses the basic construction from the proof of the corona theorem.

**Lemma 2.3.** Let \( b(z) \) be an inner function and let \( 0 < \eta < 1 \). There is a sequence \( \Gamma_i \) of disjoint rectifiable Jordan curves bounding domains \( D_i \subset D \) such that:

\[
\{ |b(z)| < \eta \} \subset \bigcup D_i. \tag{2.1}
\]

\[
\inf_{D_i} |b(z)| < \eta. \tag{2.2}
\]

(2.3) There is \( \delta = \delta(\eta) < 1 \) such that \( \Gamma_i \subset \{ |b(z)| < \delta(\eta) \} \).

(2.4) Arc length in \( \Gamma = D \cap \bigcup \Gamma_i \) is a Carleson measure on \( D \).

See [1], [2] or [10] for detailed proofs of Lemma 2.3.

To conclude the proof of Theorem 1, it suffices to assume that \( f = F \) almost everywhere, where \( F \in H^\infty(G_b(b)) \) for some \( b \in \partial \mathbb{D} \) and some \( \delta, 0 < \delta < 1 \). Using the duality \( L^\infty/H^\infty = (H^0_1)^* \), we have for \( n = 1, 2, \ldots \)

\[
\text{dist}(f, B) < \inf_{h \in H^\infty} \|f - \bar{b}^n h\| = \sup_{g \in H^1} \left| \frac{1}{2\pi i} \int_C F(z)b^n(z)g(z) \, dz \right|.
\]

Take \( \eta > 1 - \delta \) and consider the curves \( \Gamma_i \) given by Lemma 2.2. Let

\[
\Omega_r = \{ |z| < r \} \setminus \bigcup_i \overline{D_i}, \quad r < 1.
\]
By (2.2) the region \( \Omega_r \) has finite connectivity, and since \( b(z) \) is inner, \( \Omega_r \subset G_b(b) \) by (2.1) when \( \eta > 1 - \delta \). Moreover \( \Omega_r \) has rectifiable boundary consisting of
\[
J_r = \{ |z| = r \} \cap \partial \Omega_r
\]
and
\[
K_r = \{ |z| < r \} \cap \bigcup_i \Gamma_i.
\]
By (2.3) for almost every \( e^{i\theta}, re^{i\theta} \in J_r \) when \( 1 - r \) is small. Hence by dominated convergence and Lemma 2.1
\[
\lim_{r \to 1} \int_{J_r} F(z)b^n(z)g(z) \, dz = \int_C F(z)b^n(z)g(z) \, dz.
\]
By Cauchy's theorem
\[
\int_{J_r} F(z)b^n(z)g(z) \, dz = -\int_{K_r} F(z)b^n(z)g(z) \, dz
\]
with correct orientations. But by (2.3) and (2.4)
\[
\int_{K_r} |F(z)||b^n(z)||g(z)| \, ds \leq \sup|F(z)|(\delta(\eta))'\|g\|_1,
\]
where \( M \) depends only on \( \Gamma \). Sending \( n \to \infty \) completes the proof.

The theorem or its proof shows that \( \text{dist}(f, B) \) is the infimum of those \( \epsilon > 0 \) for which the condition of the theorem remains true.

3. The second characterization of \( B \) involves the distances from \( f \) to \( H^2 \), measured in the Hilbert spaces \( L^2(P_\sigma(\theta - \theta_0)d\theta) \) for points \( z_0 = r_0e^{i\theta_0} \) lying in some region \( G_b(b), b \in B \).

For \( f \in L^\infty \), let
\[
d\mu_f = |\partial f/\partial \bar{z}|^2(1 - |z|) \, dx \, dy
\]
where \( \partial/\partial \bar{z} = \frac{1}{2}(\partial/\partial x + \partial i/\partial y) \). The Littlewood-Paley identity
\[
\frac{1}{\pi} \int \int |\nabla f(z)|^2 \log \frac{1}{|z|} \, dx \, dy = \frac{1}{2\pi} \int |f - f(0)|^2 \, d\theta,
\]
where \( |\nabla f|^2 = |\partial f/\partial x|^2 + |\partial f/\partial y|^2 \), implies that \( d\mu_f \) is a finite measure on \( D \).

**Theorem 3.1.** When \( f \in L^\infty \) the following conditions are equivalent.

(i) \( f \in B \).

(ii) For any \( \epsilon > 0 \) there is \( b \in \overline{B} \) and \( \delta, 0 < \delta < 1 \), such that for all \( z_0 \in G_b(b) \),
\[
\inf_{g \in H^2} \frac{1}{2\pi} \int |f - g|^2 P_{r_0}(\theta - \theta_0) \, d\theta < \epsilon. \tag{3.1}
\]

(iii) For any \( \epsilon > 0 \) there is \( b \in \overline{B} \) and \( \delta, 0 < \delta < 1 \), such that
\[
\sup_l \frac{\mu_f(G_b(b) \cap \mathbb{R}(I))}{|I|} < \epsilon. \tag{3.2}
\]
Proof. We show (i) \(\Rightarrow\) (ii) \(\Rightarrow\) (iii) \(\Rightarrow\) (i).

Assume (i) holds. Then there is \(b \in \mathbb{R}\) and \(h \in H^\infty\) such that \(\|f - \bar{b} h\|_\infty < \varepsilon\). For \(z_0 \in G_\delta(b)\), let \(g(z) = \bar{b}(z_0) h(z)\). Then \(g \in H^2\) and
\[
\frac{1}{2\pi} \int |b h - g|^2 P_{\theta_0}(\theta - \theta_0) d\theta < \frac{\|h\|_\infty}{2\pi} \int |b(\theta) - b(z_0)|^2 P_{\theta_0}(\theta - \theta_0) d\theta
\]
\[
= \|h\|_\infty (1 - |b(z_0)|^2) < 2\delta\|h\|_\infty.
\]
Consequently, (3.1) holds if \(\delta\) is sufficiently small.

Now suppose (ii) holds and choose \(b \in \mathbb{R}\) and \(\delta\) so that (3.1) holds. We follow the proof of Lemma 2 of [3]. By Lemma 5 of [3], (3.2) will be proved if we show that
\[
\mu_f(\mathcal{R}(I_0)) < \varepsilon |I_0|
\]
for all arcs \(I_0\) of the form \(|\theta - \theta_0| < 1 - r_0\) where \(z_0 = r_0 e^{i\theta_0} \in G_\delta(b)\). Let \(w = (z - z_0)/(1 - \bar{z}_0 z)\) and let \(F(w) = f(z) - g(z)\), where \(g \in H^2\) is chosen to attain the infimum (3.1). Then \(F(w)\) is conjugate analytic, so that \(\|\nabla F(w)\|^2 = 2|\partial F/\partial \bar{w}|^2\). Since \(F(0) = 0\), the Littlewood-Paley identity gives us
\[
\frac{1}{2\pi} \int_0^{2\pi} |F(e^{i\varphi})|^2 d\varphi = \frac{2}{\pi} \iint \left| \frac{\partial F}{\partial \overline{w}} \right|^2 \log \left| \frac{1}{|w|} \right| du dv
\]
where \(w = u + iv\). A change of variables then yields
\[
\frac{1}{2\pi} \int_0^{2\pi} |f(\theta) - g(\theta)|^2 P_{\theta_0}(\theta - \theta_0) d\theta
\]
\[
= \frac{2}{\pi} \iint \left| \frac{\partial f}{\partial \overline{z}} \right|^2 \log \left| \frac{1 - \overline{z}_0 z}{z - z_0} \right| dx dy,
\]
as \(\partial g/\partial \bar{z} = 0\). When \(z \in \mathcal{R}(I_0)\),
\[
\frac{1 - |z|}{1 - r_0} < c \log \left| \frac{1 - \overline{z}_0 z}{z - z_0} \right|
\]
and hence (3.1) implies that
\[
\mu_f(\mathcal{R}(I_0)) < \frac{c(1 - r_0)}{2\pi} \int |f - g|^2 P_{\theta_0}(\theta - \theta_0) d\theta < c\varepsilon (1 - r_0).
\]

Now assume (iii). Let \(\varepsilon > 0\) and fix \(b \in \mathbb{R}\) and \(\delta, 0 < \delta < 1\), so that (3.2) holds. We estimate
\[
\text{dist}(f, \bar{b}^* H^\infty) = \sup_{g \in H^1} \left\| \frac{1}{2\pi} \int f b^i g d\theta \right\|
\]
as in the proof of Theorem 6 of [3] with one small modification. Note that when \(g \in H^1\),
\[
\nabla f(z) \nabla (b^i g)(z) = f_z(b^i g)_z + f_z(b^i g)_z = 2(\partial f/\partial \bar{z})(\partial (b^i g)/\partial z).
\]
Polarization of the Littlewood-Paley identity then yields
From this point one can repeat the proof of Theorem 6 in [3], using (3.2) instead of the analogous condition on $|\nabla f|^2(1 - |z|) \, dx \, dy$, and obtain

$$\text{dist}(f, \mathcal{B} H^\infty) \leq C e^{1/2}.$$ 

Thus $f \in B$ if (iii) holds and the theorem is proved.

The proof of the theorem contains the following estimates on $\text{dist}(f, B)$ for $f \in L^\infty$. Let $\epsilon_1(f)$ be the infimum of those $\epsilon > 0$ for which condition (ii) is true and let $\epsilon_2(f)$ be the infimum of those $\epsilon > 0$ for which condition (iii) is true. Then

$$\text{dist}(f, B) > c_1\epsilon_1^{1/2} > c_2\epsilon_2^{1/2} > c_3 \text{dist}(f, B),$$

for universal constants $c_1$, $c_2$ and $c_3$. (These inequalities, reading from the left, follow from the proofs of (i) $\Rightarrow$ (ii), (ii) $\Rightarrow$ (iii), (iii) $\Rightarrow$ (i) respectively.)

Since $\frac{\partial f}{\partial z} = (\bar{\partial} f \, \bar{\partial} z)$, the description of $B \cap \overline{B}$ given as Theorem 8 in [3] is an immediate corollary of Theorem 3.1.

**Corollary 3.2.** If $f \in L^\infty$, then $f \in B$ if and only if for any $\epsilon > 0$ there is $b \in \mathcal{B}$ and $\delta$, $0 < \delta < 1$, such that

$$\int \int_{\mathcal{D}(I) \cap G_\delta(b)} |\nabla f|^2(1 - |z|) \, dx \, dy < \epsilon(I)$$

for every arc $I$.

**References**


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