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WENDROFF TYPE INEQUALITIES

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ABSTRACT. Results (1) and (3) of Wendroff on two-variable scalar integral inequalities, quoted without proof by Beckenback and Bellman [1, p. 154] have been generalized to cover a system of integral inequalities by following an approach different from that of Jagdish Chandra and Davis [3].

In October 1976 Jagdish Chandra and Davis [3] published their results on the generalization of the Gronwall inequality to cover a system of n integral inequalities in m independent variables. At about the same time we prepared a paper in which we obtained the result in Corollary 1 in their paper by following the approach of differential analysis as in Snow [5]. Although our approach requires the hypothesis of differentiability not needed in [3], it is a constructive approach and it has a potential for being applicable to a large class of differential and integral inequalities. To illustrate this approach, we give a theorem from our unpublished paper which is a generalization of the results (1) and (3) of Wendroff, Beckenback and Bellman [1]. An application of our result is also mentioned.

THEOREM. *Let $\phi(x,y)$ be a continuous, nonnegative, n-vector function on a two dimensional domain D and $A(x,y)$, $B(x,y)$, $H(x,y)$ be continuous, nonnegative, symmetric $n \times n$ matrix functions with $A(x,y)$, $B(x,y)$ continuously differentiable in x and y and nonincreasing in y and x respectively. If C is any nonnegative constant n-vector and $\phi(x,y)$ satisfies*

$$\phi(x,y) \leqslant C + \int_0^x A(s,y)\phi(s,y)\,ds + \int_0^y B(x,t)\phi(x,t)\,dt + \int_0^x \int_0^y H(s,t)\phi(s,t)\,ds\,dt, \tag{1}$$

then

$$\phi(x,y) \leqslant C^T \exp\left[\int_0^x A(\alpha,y)\,d\alpha + \int_0^y B(x,\beta)\,d\beta + \int_0^x \int_0^y \{A(\alpha,\beta)B(\alpha,\beta) + H(\alpha,\beta)\}\,d\alpha\,d\beta\right], \tag{2}$$

where C^T is the row vector.

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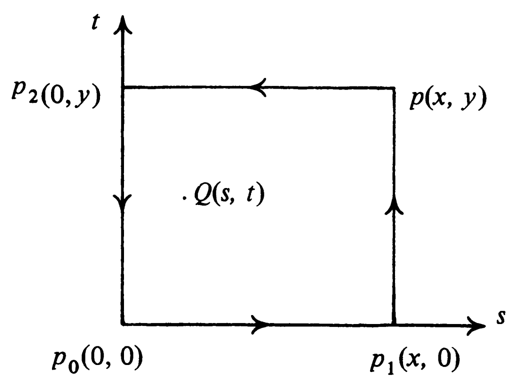
PROOF. Let $u(x, y)$ be the solution of the integral equation corresponding to (1). The existence of u can be proved by using the contraction mapping principle, as in Ghoshal and Masood [2 (see Lemma 1)]. Then by Theorem IIIb, p. 130 of Walter [6], we have

$$\phi(x, y) \leqslant u(x, y). \tag{3}$$

Differentiating $u(s, t)$ twice, we obtain

$$Lu = u_{st} - Bu_s - Au_t - (A_t + B_s + H)u = 0. \tag{4}$$

The equality (4) can be integrated using Riemann's method. For any twice continuously differentiable matrix function $W(s, t; x, y)$, we have as in Snow [5], (notation as in the figure)



$$W^T(p)u(p) = u^T(p_0)W(p_0) + \int_{p_2}^{p} [W^TA + W_s^T]u\,ds$$
$$+ \int_{p_1}^{p} u^T[BW + W_t]\,dt + \int_0^x \int_0^y [W^TLu - u^TMW]\,ds\,dt, \tag{5}$$

where M, defined by

$$MW = W_{st} + BW_s + AW_t - HW$$

is the adjoint of the operator L.

We choose W so that

(i) $W(s, t; x, y) > 0$ and $MW \geqslant 0, 0 \leqslant s \leqslant x, 0 \leqslant t \leqslant y$,

(ii) $W^TA + W_s^T \leqslant 0, 0 \leqslant s \leqslant x, t = y$,

(iii) $BW + W_t \leqslant 0, 0 \leqslant t \leqslant y, s = x$.

Then (4) and (5) imply that

$$W^T(p)u(p) \leqslant u^T(p_0)W(p_0). \tag{6}$$

A function $W(s, t; x, y)$ satisfying all the requirements (i), (ii) and (iii) is easily obtained. It is

$$W(s, t; x, y) = \exp\left[\int_s^x A(\alpha, t)\, d\alpha + \int_t^y B(s, \beta)\, d\beta + \int_s^x \int_t^y \{A(\alpha, \beta)B(\alpha, \beta) + H(\alpha, \beta)\}\, d\alpha\, d\beta\right]. \quad (7)$$

The desired conclusion now follows from (6) and (3).

The theorem can be applied to prove the uniqueness of the solution of the scalar hyperbolic differential equation

$$\phi_{xy} = \{a(x, y)\phi(x, y)\}_y + \{b(x, y)\phi(x, y)\}_x + h(x, y)f(x, y, \phi),$$

with conditions prescribed on $x = 0, y = 0$ in such a way that the problem is equivalent to the integral equation

$$\phi(x, y) = C + \int_0^x a(s, y)\phi(s, y)\, ds + \int_0^y b(x, t)\phi(x, t)\, dt + \int_0^x \int_0^y h(s, t)f(s, t, \phi(s, t))\, ds\, dt;$$

where C is a nonnegative constant.

Of course the functions ϕ, a, b, h must be continuous nonnegative with a, b continuously differentiable and nonincreasing in y and x respectively and f must be Lipschitzian in its third argument. (See Ghoshal and Masood [**2**].) Note that Theorem 11.2.1 in Lakshmikantham and Leela [**4**] is not applicable in this situation. Further the applicability of the uniqueness theorem in Walter [**6**, p. 166] is also not immediately obvious for the problem.

References

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