WENDROFF TYPE INEQUALITIES

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Abstract. Results (1) and (3) of Wendroff on two-variable scalar integral inequalities, quoted without proof by Beckenback and Bellman [1, p. 154] have been generalized to cover a system of integral inequalities by following an approach different from that of Jagdish Chandra and Davis [3].

In October 1976 Jagdish Chandra and Davis [3] published their results on the generalization of the Gronwall inequality to cover a system of \( n \) integral inequalities in \( m \) independent variables. At about the same time we prepared a paper in which we obtained the result in Corollary 1 in their paper by following the approach of differential analysis as in Snow [5]. Although our approach requires the hypothesis of differentiability not needed in [3], it is a constructive approach and it has a potential for being applicable to a large class of differential and integral inequalities. To illustrate this approach, we give a theorem from our unpublished paper which is a generalization of the results (1) and (3) of Wendroff, Beckenback and Bellman [1]. An application of our result is also mentioned.

Theorem. Let \( \phi(x, y) \) be a continuous, nonnegative, \( n \)-vector function on a two dimensional domain \( D \) and \( A(x, y), B(x, y), H(x, y) \) be continuous, nonnegative, symmetric \( n \times n \) matrix functions with \( A(x, y), B(x, y) \) continuously differentiable in \( x \) and \( y \) and nonincreasing in \( y \) and \( x \) respectively. If \( C \) is any nonnegative constant \( n \)-vector and \( \phi(x, y) \) satisfies

\[
\phi(x, y) \leq C + \int_0^x A(s, y)\phi(s, y) \, ds + \int_0^y B(x, t)\phi(x, t) \, dt \\
\quad + \int_0^x \int_0^y H(s, t)\phi(s, t) \, ds \, dt,
\]

then

\[
\phi(x, y) \leq C^T \exp \left[ \int_0^x A(\alpha, y) \, d\alpha + \int_0^y B(\alpha, \beta) \, d\beta \\
\quad + \int_0^x \int_0^y \{ A(\alpha, \beta)B(\alpha, \beta) + H(\alpha, \beta) \} \, d\alpha \, d\beta \right],
\]

where \( C^T \) is the row vector.

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PROOF. Let \( u(x, y) \) be the solution of the integral equation corresponding to (1). The existence of \( u \) can be proved by using the contraction mapping principle, as in Ghoshal and Masood [2 (see Lemma 1)]. Then by Theorem IIIb, p. 130 of Walter [6], we have

\[
\phi(x, y) \leq u(x, y).
\]  

Differentiating \( u(s, t) \) twice, we obtain

\[
Lu = u_{st} - Bu_t - Au_t - (A_s + B_s + H)u = 0.
\]  

The equality (4) can be integrated using Riemann's method. For any twice continuously differentiable matrix function \( W(s; x, y) \), we have as in Snow [5], (notation as in the figure)

\[
W^T(p)u(p) = u^T(p_0)W(p_0) + \int_{p_0}^{p} [W^T_s + W_s^T]u \, ds \\
+ \int_{p_1}^{p} u^T[BW + W_t] \, dt + \int_{0}^{s} \int_{0}^{t} [W^T L u - u^T M W] \, ds \, dt,
\]  

where \( M \), defined by

\[
M W = W_{st} + BW_s + AW_t - HW
\]
is the adjoint of the operator \( L \).

We choose \( W \) so that

(i) \( W(s, t; x, y) > 0 \) and \( MW > 0 \), \( 0 < s < x, 0 < t < y \),

(ii) \( W^T_A + W_s^T < 0 \), \( 0 < s < x, t = y \),

(iii) \( BW + W_t < 0 \), \( 0 < t < y, s = x \).

Then (4) and (5) imply that

\[
W^T(p)u(p) \leq u^T(p_0)W(p_0).
\]  

A function \( W(s, t; x, y) \) satisfying all the requirements (i), (ii) and (iii) is easily obtained. It is
\[ W(s, t; x, y) = \exp \left[ \int_s^x A(\alpha, t) \, d\alpha + \int_t^y B(s, \beta) \, d\beta \right. \\
\left. + \int_s^x \int_t^y \{ A(\alpha, \beta)B(\alpha, \beta) + H(\alpha, \beta) \} \, d\alpha \, d\beta \right]. \quad (7) \]

The desired conclusion now follows from (6) and (3).

The theorem can be applied to prove the uniqueness of the solution of the scalar hyperbolic differential equation

\[ \phi_{xy} = \{ a(x, y)\phi(x, y) \}_y + \{ b(x, y)\phi(x, y) \}_x + h(x, y)f(x, y, \phi), \]

with conditions prescribed on \( x = 0, y = 0 \) in such a way that the problem is equivalent to the integral equation

\[ \phi(x, y) = C + \int_0^x a(s, y)\phi(s, y) \, ds + \int_0^y b(x, t)\phi(x, t) \, dt \\
+ \int_0^x \int_0^y h(s, t)f(s, t, \phi(s, t)) \, ds \, dt; \]

where \( C \) is a nonnegative constant.

Of course the functions \( \phi, a, b, h \) must be continuous nonnegative with \( a, b \) continuously differentiable and nonincreasing in \( y \) and \( x \) respectively and \( f \) must be Lipschitzian in its third argument. (See Ghoshal and Masood [2].) Note that Theorem 11.2.1 in Lakshmikantham and Leela [4] is not applicable in this situation. Further the applicability of the uniqueness theorem in Walter [6, p. 166] is also not immediately obvious for the problem.

**References**


