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THE QUASI-MULTIPLIER CONJECTURE

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ABSTRACT. It is shown by example that the left and right multipliers of a C^*-algebra do not always span the linear space of quasi-multipliers.

Let A be the realization of a C^*-algebra as a family of operators on a Hilbert space H, and let $VN(A)$ be the closure of A in the weak operator topology. Write $LM(A)$, $RM(A)$, and $QM(A)$ respectively for the families of all elements $v \in VN(A)$ such that respectively $vA \subset A$, $Av \subset A$, and $AvA \subset A$. Then $QM(A)$ may be identified with the family of all quasi-multipliers on A (see **[2]** and **[1**, Proposition 4.2]). Akemann and Pedersen **[1]** have raised the conjecture that $QM(A) = LM(A) + RM(A)$. We develop here a counter-example.

Let H be the Hilbert space $l_2(\mathbf{R})$. For each $r \in \mathbf{R}$, write ξ_r for the characteristic function of the singleton $\{r\}$; π_r for the orthogonal projection of H onto the subspace generated by $\{\xi_t\colon t \geqslant r\}$; and write p_r for the projection complementary to π_r. Let $B(H)$ be the set of all bounded operators on H and $B_0(H)$ the set of compact operators. For each $x \in B(H)$, let $d(x)$ be the cardinality of an orthonormal basis for $x(H)$.

Let A^- be the C^*-algebra $\{x \in B(H)\colon \max\{d(p_0x), d(xp_0)\} \leqslant \aleph_0\}$. From $B_0(H) \subset A^-$ follows that $VN(A^-) = B(H)$.

LEMMA 1. *$LM(A^-) = \{v \in B(H)\colon d(p_0v\pi_0) \leqslant \aleph_0\}$ and $LM(A^-) + RM(A^-) = QM(A^-) = B(H)$.*

PROOF. If $d(p_0v\pi_0) \leqslant \aleph_0$ and $x \in A^-$, then

$$d(p_0vx) \leqslant d(p_0v\pi_0x) + d(p_0vp_0x) \leqslant d(p_0v\pi_0) + d(p_0x) \leqslant \aleph_0$$

and

$$d(vxp_0) \leqslant d(xp_0) \leqslant \aleph_0$$

so $v \in LM(A^-)$. Conversely, if $w \in LM(A^-)$, then $w\pi_0 \in A^-$ so $d(p_0w\pi_0) \leqslant \aleph_0$.

Consider any $y \in B(H)$. Then $p_0(y\pi_0)^*\pi_0 = 0$ so $(y\pi_0)^* \in LM(A^-)$; hence $y\pi_0 \in RM(A^-)$. But clearly $yp_0 \in LM(A^-)$ so

$$y = yp_0 + y\pi_0 \in LM(A^-) + RM(A^-). \quad \text{Q.E.D.}$$

Now let A^+ be the C^*-algebra

$$\left\{x \in B(H)\colon \lim_{r\to\infty} \pi_r x = \lim_{r\to\infty} x\pi_r = 0\right\}.$$

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Again from $B_0(H) \subset A^+$ follows that $VN(A^+) = B(H)$.

LEMMA 2. *Let* $X = \{v \in B(H)\colon (\forall t \in \mathbf{R}) \lim_{r\to\infty} \pi_r v p_t = 0\}$. *Then* $X = LM(A^+)$ *and* $LM(A^+) + RM(A^+) = B(H) = QM(A^+)$.

PROOF. Consider $v \in X$ and $x \in A^+$. Obviously $\lim_{r\to\infty} vx\pi_r = 0$. Further

$$\begin{aligned}\varlimsup_{r\to\infty} \|\pi_r v x\| &\leqslant \varlimsup_{t\to\infty} \varlimsup_{r\to\infty} (\|\pi_r v \pi_t x\| + \|\pi_r v p_t x\|)\\ &\leqslant \varlimsup_{t\to\infty} \|v\|\, \|\pi_t x\| + \varlimsup_{t\to\infty} \varlimsup_{r\to\infty} \|\pi_r v p_t\|\, \|x\| = 0.\end{aligned}$$

Hence, $x \in LM(A^+)$. Conversely, for $w \in LM(A^+)$ and $t \in \mathbf{R}$, both p_t and wp_t are in A^+ so evidently $w \in X$.

Consider any $y \in B(H)$. Define v and w in $B(H)$ by letting $v(\xi_u) = p_u y(\xi_u)$ and $w(\xi_u) = \pi_u y(\xi_u)$ for all $u \in \mathbf{R}$. Then $y = v + w$. Furthermore, for $t, r \in \mathbf{R}$ such that $r > t$, simple calculations show that $\pi_r v p_t = 0$ and $p_t w \pi_r = 0$. It follows that $v, w^* \in X$; hence $v \in LM(A^+)$ and $w \in RM(A^+)$. Q.E.D.

Now let $A = A^- \cap A^+$. Again $B_0(H) \subset A$ so $VN(A) = B(H)$.

LEMMA 3. $LM(A) = LM(A^-) \cap LM(A^+)$.

PROOF. That $LM(A^-) \cap LM(A^+) \subset LM(A)$ is trivial. Let $w \in LM(A)$ be arbitrary.

Assume $d(p_0 w \pi_0) > \aleph_0$. Then, for some $n > 0$, $d(p_0 w x_0 p_n) > \aleph_0$ as well. Evidently $\pi_0 p_n$ is in A, so that $w\pi_0 p_n$ is in A as well. In particular $w\pi_0 p_n$ is in A^- so $d(p_0 w \pi_0 p_n) < \aleph_0$: an absurdity. It follows by Lemma 1 that $w \in LM(A^-)$.

Assume that, for some $t \in \mathbf{R}$, $\lim_{r\to\infty} \pi_r w p_t \neq 0$. Then, for some $\varepsilon > 0$ and each natural number n, there exist finite subsets $\{r(n;j)\}_{j=1}^{m(n)}$ and $\{s(n;j)\}_{j=1}^{m(n)}$ of R such that

$$|\alpha_n| = 1 \quad \text{and} \quad |\pi_n w p_t(\alpha_n)| \geqslant \varepsilon$$

where $\alpha_n = \sum_{j=1}^{m(n)} r(n;j)\xi_{s(n;j)}$. Let τ be the orthogonal projection onto the subspace of H generated by the set of all the vectors $\xi_{s(n;j)}$. Evidently $\tau p_t = p_t \tau$ is in A. Hence $wp_t\tau$ is in A and, in particular, in A^+ as well. Thus

$$0 = \lim_n \|\pi_n w p_t \tau\| \geqslant \varlimsup_n |\pi_n w p_t \tau(\alpha_n)| \geqslant \varepsilon$$

since $\tau(\alpha_n) = \alpha_n$: an absurdity. It follows by Lemma 2 that $w \in LM(A^+)$. Q.E.D.

We note that, since evidently $QM(A^-) \cap QM(A^+) \subset QM(A)$, we have $QM(A) = B(H)$.

THEOREM. $LM(A) + RM(A) \neq QM(A)$.

PROOF. Let $s \in QM(A)$ be the partial isometry: $s(\xi_t) = 0$ for $t \leqslant 0$ and $s(\xi_t) = \xi_{-t}$ for $t \geqslant 0$. Assume $s = v + w$ for $v \in LM(A)$ and $w \in RM(A)$. Then $w^* \in LM(A)$ so, by Lemma 2,

$$0 = \lim_{r\to\infty} \pi_r w^* p_0 \quad \text{so} \quad \lim_{r\to\infty} p_0 w \pi_r = 0 \text{ as well.}$$

Choose $r > 0$ such that $\|p_0 w \pi_r\| < 1/2$. Since $s = p_0 s \pi_0$, we have

$$s\pi_r = p_0 s \pi_r = p_0 v \pi_r + p_0 w \pi_r.$$

From Lemma 1, $d(p_0 v \pi_r) \leqslant \aleph_0$. Thus $\|s\pi_r - y\| \leqslant 1/2$ for some operator y with countable rank: an absurdity. Q.E.D.

References

1. Charles A. Akemann and Gert K. Pederson, *Complications of semi-continuity in C*-algebra theory*, Duke Math. J. **40** (1973), 785–795.

2. Kelly McKennon, *Quasi-multipliers*, Trans. Amer. Math. Soc. **233** (1977), 105–123.

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