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# SYMMETRIC AND ORDINARY DIFFERENTIATION

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ABSTRACT. In 1927, A. Khintchine proved that a measurable symmetrically differentiable function $f$ mapping the real line $R$ into itself is differentiable in the ordinary sense at each point of $R$ except possibly for a set of Lebesgue measure zero. Here it is shown that this exceptional set is also of the first Baire category; even more, it is shown to be a $\sigma$-porous set of E. P. Dolženko.

**1. Introduction.** Let $f$ be a real valued function defined on the real line $R$. The *upper symmetric derivate of f at* $x \in R$ is

$$D^s f(x) \equiv \limsup_{h \to 0} \frac{f(x+h) - f(x-h)}{2h}.$$

The *lower symmetric derivate* $D_s f(x)$ *of f at x* is the corresponding limit inferior. If $D^s f(x) = D_s f(x)$, then this common value is called the *symmetric derivative of f at x* and is denoted $f^s(x)$.

It is readily observed that the existence of the ordinary derivative $f'(x)$ implies the existence of $f^s(x)$, with the two values being equal. Concerning the reverse implication, Khintchine [4] proved: *a measurable function* $f$: $R \to R$ *has a finite derivative* $f'(x)$ *at almost every point where* $D^s f(x) < \infty$. We observe that the exceptional set in this theorem, although of Lebesgue measure zero, need not be of the first Baire category even if the condition $D^s f(x) < \infty$ is replaced by the stronger condition that $f^s(x)$ exists and is finite. To see this, consider $f$ to be the characteristic function of an additive group $G$ of real numbers that is of measure zero and of the second category. Then $f^s(x) = 0$ for each $x \in G$, but $f'(x)$ exists at no point of $G$. We also observe that the exceptional set in Khintchine's theorem need not be of the first category when strong continuity conditions are placed on the function. Indeed, let $Z$ be any bounded subset of $R$ that is both of measure 0 and of the second category. According to C. Goffman [3], there exists a bounded measurable subset $S$ of $R$ whose metric density does not exist at any point of $Z$. Let $\chi_S$ be the characteristic function of $S$ and set

$$f(x) = \int_{-\infty}^{x} \chi_S(t)\, dt.$$

Then $f$ is a Lipschitz function (with its symmetric derivates bounded in

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absolute value by 1 everywhere), but $f'(x)$ does not exist at any point of $Z$.

With the two examples of the previous paragraph in mind, we conclude that in order to obtain a category analogue of Khintchine's theorem we must simultaneously impose some sort of continuity conditions on $f$ and replace the condition $D^s f(x) < \infty$ by the stronger condition that $f^s(x)$ exists. In so doing, our result (Theorem 3 in §6) says that if $f$ belongs to the function class

$$\mathfrak{F} \equiv \{f\colon \text{the points of continuity of } f \text{ are dense in } R\},$$

then the existence of $f^s(x)$ implies the existence of $f'(x)$ at all but a $\sigma$-porous set of points (see §2 for the definition of these sets).

This theorem is a direct consequence of Theorem 2 in §5, which gives a relationship between the symmetric derivates and the Dini derivates of functions $f$ in $\mathfrak{F}$. The proof of Theorem 2 rests on the determination of the size of the difference set $\mathfrak{I}^* - \mathfrak{I}$, where

$$\mathfrak{I}^* = \{x\colon f(x-h) \leqslant f(x+h) \text{ for sufficiently small } h > 0\}$$

and

$$\mathfrak{I} = \{x\colon f(x-h) \leqslant f(x) \leqslant f(x+h) \text{ for sufficiently small } h > 0\}.$$

This determination is made in §4, while §§2 and 3 are devoted to the establishment of some preliminary results that will be used there.

**2. Porosity lemmas.** The notion of porosity is due to E. P. Dolženko [**1**]. The *porosity of the set $S \subset R$ at the point $x \in R$* is defined to be the nonnegative value

$$\limsup_{r\downarrow 0} \frac{l(x,r,S)}{r},$$

where $l(x, r, S)$ denotes the length of the largest open interval contained in the set $(x-r, x+r) \cap (R-S)$. The set $S$ is called *porous* if it has positive porosity at each of its points, and it is called *$\sigma$-porous* if it is a countable union of porous sets. It follows readily from the definition that a $\sigma$-porous set is both of the first category and of measure zero. On the other hand, L. Zajíček [**5**] has constructed a perfect set of measure zero that is not $\sigma$-porous.

We now introduce some terminology that will be used in the statements of the following two lemmas concerning points at which a set has porosity $< 1/2$: Let $I$ be an open interval and let $s$ be a point of $R$. The *reflection of $I$ in $s$* is defined to be the open interval $\{2s - x\colon x \in I\}$; it is called a *left* [resp., *right*] reflection of $I$ in $s$ if $s \leqslant (a+b)/2$ [resp., $s \geqslant (a+b)/2$], where $I \equiv (a, b)$. If $S \subset R$, then an open interval $I'$ is said to be a *finite left* [resp., *right*] *reflection of $I$ in $S$* if there exists a finite collection $\{I_1, I_2, \ldots, I_k\}$ of open intervals and a corresponding subset $\{s_1, s_2, \ldots, s_{k-1}\}$ of $S$ such that $I = I_1$, $I' = I_k$, and $I_{j+1}$ is the left [resp., right] reflection of $I_j$ in $s_j$ for $j = 1, 2, \ldots, k-1$. Analogously we define what is meant by a point $p'$ being a finite left or right reflection of a point $p$ in $S$.

LEMMA 1. *If $S \subset R$ and $l(0, r, S)/r < 1/2$ for $0 < r < \delta$, then each open interval $(a, b)$ contained in $(0, \delta)$ [resp., $(-\delta, 0)$] has a finite left [resp., right] reflection in $S$ that contains* 0.

PROOF. Suppose $I \equiv (a, b) \subset (0, \delta)$. If $a = 0$, then the reflection of $I$ in any point of $S \cap (0, b/2)$ contains 0. Suppose $a \neq 0$. It follows from the hypothesis that there exists a point $s$ in $S \cap (a/2, a)$, and clearly the reflection $(a', b')$ of $(a, b)$ in $s$ is such that $b' > 0$. If $0 \notin [a', b')$, then reflect $(a', b')$ in some point $s'$ in $S \cap (a'/2, a')$ to obtain an interval $(a'', b'')$ with $b'' > 0$. Since each of the reflected intervals has the same length as $I$, a finite repetition of this process will produce a finite left reflection $(a^*, b^*)$ of $I$ in $S$ with $0 \in [a^*, b^*)$ and $b^* > 0$. Either $(a^*, b^*)$ contains 0 or $a^* = 0$ and the reflection of $(a^*, b^*)$ in any point of $S \cap (0, b^*/2)$ contains 0. Thus the lemma is established in the case $I \subset (0, \delta)$, and the case $I \subset (-\delta, 0)$ is handled similarly.

LEMMA 2. *If $S \subset R$ and $l(0, r, S)/r < \lambda < 1/2$ for $0 < r < \delta$, then, for every $\varepsilon > 0$, each point $p$ in $(0, \delta)$ [resp., $(-\delta, 0)$] has a finite left [resp., right] reflection in $S$ that is contained in $(0, \varepsilon)$ [resp., $(-\varepsilon, 0)$].*

PROOF. Suppose $p \in (0, \delta)$. It follows from the hypothesis that there exists a point $s$ in $S \cap (p/2, p/2 + \lambda p)$. If $p_1$ is the reflection of $p$ in $s$, then

$$0 < p_1 = 2s - p < 2\lambda p.$$

Similarly we can reflect $p_1$ in some point $s_1$ in $S \cap (p_1/2, p_1/2 + \lambda p_1)$ to obtain a point $p_2$ with

$$0 < p_2 < 2\lambda p_1 < (2\lambda)^2 p.$$

Continuing this reflection process inductively, we obtain a sequence $p_1, p_2, \ldots$ of points with $0 < p_n < (2\lambda)^n p$ for each index $n$. Then, since $0 < \lambda < 1/2$, it follows that $p_n \downarrow 0$. This establishes the lemma for $p \in (0, \delta)$, and the proof for $p \in (-\delta, 0)$ is similar.

**3. Porosity lemmas applied to the sets $S_n$.** In the next section, we will prove that the set $\mathcal{S}^* - \mathcal{S}$ is a $\sigma$-porous set for functions $f \in \mathcal{F}$, and we will find it convenient to use the decomposition $\mathcal{S}^* = \bigcup_{n=1}^{\infty} S_n$ where

$$S_n = \{x : f(x - h) \leqslant f(x + h) \text{ for } 0 < h < 1/n\}.$$

In that proof, we make use of certain properties that an arbitrary function $f$: $R \to R$ possesses at a point where the porosity of $S_n$ is $< 1/2$. These properties are given in the next two lemmas.

LEMMA 3. *Let $f$: $R \to R$ be arbitrary, and let $\mathcal{C}(f)$ denote the set of points where $f$ is continuous. If $l(0, r, S_n)/r < 1/2$ for $0 < r < \delta < 1/n$, then*

$$\sup_{\substack{-\delta < x < 0 \\ x \in \mathcal{C}(f)}} f(x) \leqslant \underline{\lim}_{x \to 0} f(x) \leqslant f(0) \leqslant \overline{\lim}_{x \to 0} f(x) \leqslant \inf_{\substack{0 < x < \delta \\ x \in \mathcal{C}(f)}} f(x). \qquad (*)$$

PROOF. By hypothesis, there exists a strictly decreasing sequence $s_1, s_2, \ldots$ of points in $S_n \cap (0, \delta)$ that converges to 0. Since $s_k \in S_n$, we have $f(0) \leqslant f(2s_k)$ which yields the third inequality in (*). A similar proof establishes the second inequality in (*).

Now choose a point $c \in \mathcal{C}(f) \cap (0, \delta)$ and let $\varepsilon > 0$ be given. By the continuity of $f$ at $c$, there is an open subinterval $I$ of $(0, \delta)$ such that $f(x) < f(c) + \varepsilon$ for each $x \in I$. According to Lemma 1, there is a finite left reflection $I'$ of $I$ in $S_n$ with $0 \in I'$. Since $I'$ is a left reflection of $I$ in $S_n$, we have $f(x) < f(c) + \varepsilon$ for each $x \in I'$. The last inequality in (*) now follows readily, and the first inequality in (*) is established in a similar manner. This completes the proof of the lemma.

LEMMA 4. *Let $f$: $R \to R$ be arbitrary. If $l(0, r, S_n)/r < \lambda < 1/2$ for $0 < r < \delta < 1/n$, then*

$$\varliminf_{x \to 0^+} f(x) = \inf_{0<x<\delta} f(x) \quad \text{and} \quad \varlimsup_{x \to 0^-} f(x) = \sup_{-\delta<x<0} f(x). \qquad (\dagger)$$

PROOF. Set $\alpha = \inf_{0<x<\delta} f(x)$ and let $\varepsilon > 0$ be given. Choose $x \in (0, \delta)$ for which $f(x) < \alpha + \varepsilon$. By Lemma 2 there exists a sequence $x = x_1, x_2, \ldots$ of points such that $x_j \to 0$ and $x_{j+1}$ is a left reflection of $x_j$ in $S_n$ for each index $j$. Hence $f(x_{j+1}) \leqslant f(x_j)$ for each index $j$ and it follows that $\varliminf_{j\to\infty} f(x_j) < \alpha + \varepsilon$. This proves the first equality in (†); the proof of the second equality is similar, and the lemma is established.

**4. The set $\mathcal{G}^* - \mathcal{G}$.**

THEOREM 1. *If $f \in \mathcal{F}$, then $\mathcal{G}^* - \mathcal{G}$ is σ-porous; indeed, it is a countable union of sets, each of which has porosity $\geqslant 1/2$ at each of its points.*

PROOF. Let $H_n$ denote the set of all points in $S_n - \mathcal{G}$ at which $S_n$ has porosity $< 1/2$, and choose a point $\hat{x} \in H_n$. Let $\lambda$ and $\delta$ be numbers such that $l(\hat{x}, r, S_n)/r < \lambda < 1/2$ for $0 < r < \delta < 1/n$. Replacing 0 by $\hat{x}$ in Lemma 3, we have

$$\sup_{\substack{0<h<\delta \\ \hat{x}-h \in \mathcal{C}(f)}} f(\hat{x} - h) \leqslant \inf_{\substack{0<h<\delta \\ \hat{x}+h \in \mathcal{C}(f)}} f(\hat{x} + h). \qquad (')$$

However, if equality holds in ('), it follows from Lemma 3 that $f(x)$ is continuous at $\hat{x}$. So, replacing 0 by $\hat{x}$ in Lemma 4, we have that

$$\sup_{0<h<\delta} f(\hat{x} - h) = f(\hat{x}) = \inf_{0<h<\delta} f(\hat{x} + h),$$

that is, $\hat{x} \in \mathcal{G}$. But this contradicts $\hat{x} \in H_n$ and so the strict inequality must hold in (').

Now for each rational number $\alpha$ and each positive integer $k$ let $A_{\alpha k}$ denote the set of all points $x \in R$ for which

$$\sup_{\substack{0<h<1/k \\ x-h \in \mathcal{C}(f)}} f(x - h) < \alpha < \inf_{\substack{0<h<1/k \\ x+h \in \mathcal{C}(f)}} f(x + h).$$

From the discussion in the previous paragraph, it follows that $H_n$ is a subset of the countable union of all the sets $A_{\alpha k}$; furthermore, since $\mathcal{C}(f)$ is dense in $R$, it is easily observed that each of the sets $A_{\alpha k}$ is an isolated set, more specifically, if $x \in A_{\alpha k}$ then $A_{\alpha k} \cap (x - 2/k, x + 2/k) = \{x\}$. Hence $H_n$ is a countable set, and we can write

$$S_n - \mathcal{G} = \left[(S_n - \mathcal{G}) - H_n\right] \cup \left( \bigcup_{x \in H_n} \{x\} \right),$$

where each set in the union on the right has porosity $\geqslant 1/2$ at each of its points. This proves the theorem.

**5. Dini derivates and symmetric derivates.** The *upper right* and the *upper left Dini derivates* of the function $f\colon R \to R$ are respectively

$$D^+f(x) \equiv \limsup_{h \to 0^+} \frac{f(x+h) - f(x)}{h}$$

and

$$D^-f(x) \equiv \limsup_{h \to 0^-} \frac{f(x+h) - f(x)}{h}.$$

The *lower Dini derivates* $D_+ f(x)$ and $D_- f(x)$ are the corresponding lim inf's.

THEOREM 2. *If $f \in \mathcal{F}$, then for all but a σ-porous set of points both of the following equalities hold*:

(i) $D_s f(x) = \min\{D_+ f(x), D_- f(x)\}$,
(ii) $D^s f(x) = \max\{D^+ f(x), D^- f(x)\}$.

PROOF. We first observe that for each point $x$ we have

$$\min\{D_- f(x), D_+ f(x)\} \leqslant D_s f(x) \leqslant D^s f(x) \leqslant \max\{D^- f(x), D^+ f(x)\}. \tag{$*$}$$

Now for each rational number $\alpha$ set

$$N(f, \alpha) = \left\{x\colon D_s f(x) > \alpha > \min\{D_- f(x), D_+ f(x)\}\right\}.$$

Then, in light of the first inequality in $(*)$, we see that in order to prove the statement of the theorem concerning equality (i), it suffices to show that each set $N(f, \alpha)$ is σ-porous. Furthermore, it is sufficient to show that $N(f, 0)$ is σ-porous because $N(f, \alpha) = N(g, 0)$ for the function $g(x) = f(x) - \alpha x$. But this is a direct consequence of Theorem 1 since $N(f, 0) \subset \mathcal{G}^* - \mathcal{G}$.

Now, replacing $f(x)$ with $-f(x)$, we use this same argument to establish the statement of the theorem concerning equality (ii); hence, the proof is complete.

**6. Consequences of Theorem 2.** An immediate consequence of Theorem 2 is the following theorem which says that, for functions in class $\mathcal{F}$, the ordinary derivative exists at *most* points where the symmetric derivative exists.

THEOREM 3. *If $f \in \mathfrak{F}$, then $f'(x)$ exists at all but a $\sigma$-porous set of points where $f^s(x)$ exists.*

Now, according to the theorem of Khintchine given in §1 of this paper, a measurable symmetrically differentiable function must be differentiable in the ordinary sense almost everywhere and is therefore in class $\mathfrak{F}$. This observation together with Theorem 3 yields the following result which was announced in the abstract of this article.

THEOREM 4. *If $f$ is measurable and symmetrically differentiable on $R$, then $f'(x)$ exists for all but a $\sigma$-porous set of points.*

Although it has been known for some time that a continuous symmetrically differentiable function is differentiable in the ordinary sense at each point of $R$ except possibly for a set of the first category and measure zero, it was only recently shown that this exceptional set need not be countable. That is, based upon a construction of J. Foran [2], we see that this exceptional set can equal certain perfect sets of measure zero. In this connection, Foran [2] posed the question as to whether each perfect set of measure zero could be such an exceptional set. Theorem 4 provides a negative answer to this question; that is, the non-$\sigma$-porous perfect set of measure zero constructed by L. Zajíček [5] cannot be such an exceptional set. Nevertheless, the following specialization of Foran's question remains open:

*Is each perfect $\sigma$-porous set precisely the set of points where some continuous symmetrically differentiable function fails to be differentiable in the ordinary sense?*

The final consequence of Theorem 2 that we wish to mention concerns points of density of a measurable set. It is well known that the set of points $x$ where the symmetric metric density of a measurable set $M$ exists but the ordinary metric density does not constitutes a set of the first category and measure zero. By considering $f(x)$ to be an integral of the characteristic function of $M$, we see that Theorem 3 allows us to say more.

THEOREM 5. *If $M$ is a measurable subset of $R$, then for all but a $\sigma$-porous set of points $x \in R$ the existence of the symmetric metric density of $M$ at $x$ implies the existence of the ordinary metric density of $M$ at $x$.*

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