SYMMETRIC AND ORDINARY DIFFERENTIATION

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ABSTRACT. In 1927, A. Khintchine proved that a measurable symmetrically
differentiable function \( f \) mapping the real line \( \mathbb{R} \) into itself is differentiable
in the ordinary sense at each point of \( \mathbb{R} \) except possibly for a set of
Lebesgue measure zero. Here it is shown that this exceptional set is also of
the first Baire category; even more, it is shown to be a \( \sigma \)-porous set of E. P.
Dolženko.

1. Introduction. Let \( f \) be a real valued function defined on the real line \( \mathbb{R} \).
The upper symmetric derivative of \( f \) at \( x \in \mathbb{R} \) is
\[
D^+f(x) = \lim_{h \to 0^+} \frac{f(x + h) - f(x - h)}{2h}.
\]
The lower symmetric derivative \( D^-f(x) \) of \( f \) at \( x \) is the corresponding limit
inferior. If \( D^+f(x) = D^-f(x) \), then this common value is called the symmetric
derivative of \( f \) at \( x \) and is denoted \( f^*(x) \).

It is readily observed that the existence of the ordinary derivative \( f'(x) \)
implies the existence of \( f^*(x) \), with the two values being equal. Concerning
the reverse implication, Khintchine [4] proved: a measurable function \( f: \mathbb{R} \to \mathbb{R} \)
has a finite derivative \( f'(x) \) at almost every point where \( D^+f(x) < \infty \).
We observe that the exceptional set in this theorem, although of Lebesgue
measure zero, need not be of the first Baire category even if the condition
\( D^+f(x) < \infty \) is replaced by the stronger condition that \( f^*(x) \) exists and is
finite. To see this, consider \( f \) to be the characteristic function of an additive
group \( G \) of real numbers that is of measure zero and of the second category.
Then \( f^*(x) = 0 \) for each \( x \in G \), but \( f'(x) \) exists at no point of \( G \). We also
observe that the exceptional set in Khintchine's theorem need not be of the
first category when strong continuity conditions are placed on the function.
Indeed, let \( Z \) be any bounded subset of \( \mathbb{R} \) that is both of measure 0 and of
the second category. According to C. Goffman [3], there exists a bounded
measurable subset \( S \) of \( \mathbb{R} \) whose metric density does not exist at any point of
\( Z \). Let \( \chi_S \) be the characteristic function of \( S \) and set
\[
f(x) = \int_{-\infty}^{x} \chi_S(t) \, dt.
\]
Then \( f \) is a Lipschitz function (with its symmetric derivatives bounded in
absolute value by 1 everywhere), but \( f'(x) \) does not exist at any point of \( Z \).

With the two examples of the previous paragraph in mind, we conclude that in order to obtain a category analogue of Khintchine’s theorem we must simultaneously impose some sort of continuity conditions on \( f \) and replace the condition \( D^*f(x) < \infty \) by the stronger condition that \( f^*(x) \) exists. In so doing, our result (Theorem 3 in §6) says that if \( f \) belongs to the function class

\[
\mathcal{F} \equiv \{ f : \text{the points of continuity of } f \text{ are dense in } \mathbb{R} \},
\]

then the existence of \( f^*(x) \) implies the existence of \( f'(x) \) at all but a \( \sigma \)-porous set of points (see §2 for the definition of these sets).

This theorem is a direct consequence of Theorem 2 in §5, which gives a relationship between the symmetric derivatives and the Dini derivatives of functions \( f \) in \( \mathcal{F} \). The proof of Theorem 2 rests on the determination of the size of the difference set \( \mathcal{D} = \mathcal{F}^* - \mathcal{F} \), where

\[
\mathcal{D}^* = \{ x : f(x - h) < f(x + h) \text{ for sufficiently small } h > 0 \}
\]

and

\[
\mathcal{D} = \{ x : f(x - h) < f(x) < f(x + h) \text{ for sufficiently small } h > 0 \}.
\]

This determination is made in §4, while §§2 and 3 are devoted to the establishment of some preliminary results that will be used there.

2. Porosity lemmas. The notion of porosity is due to E. P. Dolzenko [1]. The porosity of the set \( S \subset \mathbb{R} \) at the point \( x \in \mathbb{R} \) is defined to be the nonnegative value

\[
\limsup_{r \to 0} \frac{l(x, r, S)}{r},
\]

where \( l(x, r, S) \) denotes the length of the largest open interval contained in the set \( (x - r, x + r) \cap (\mathbb{R} - S) \). The set \( S \) is called porous if it has positive porosity at each of its points, and it is called \( \sigma \)-porous if it is a countable union of porous sets. It follows readily from the definition that a \( \sigma \)-porous set is both of the first category and of measure zero. On the other hand, L. Zajiček [5] has constructed a perfect set of measure zero that is not \( \sigma \)-porous.

We now introduce some terminology that will be used in the statements of the following two lemmas concerning points at which a set has porosity < 1/2: Let \( I \) be an open interval and let \( s \) be a point of \( \mathbb{R} \). The reflection of \( I \) in \( s \) is defined to be the open interval \( (2s - x : x \in I) \); it is called a left [resp., right] reflection of \( I \) in \( s \) if \( s < (a + b)/2 \) [resp., \( s > (a + b)/2 \)], where \( I \equiv (a, b) \). If \( S \subset \mathbb{R} \), then an open interval \( I' \) is said to be a finite left [resp., right] reflection of \( I \) in \( S \) if there exists a finite collection \( \{ I_1, I_2, \ldots, I_k \} \) of open intervals and a corresponding subset \( \{ s_1, s_2, \ldots, s_{k-1} \} \) of \( S \) such that \( I = I_1, I' = I_k, \) and \( I_{j+1} \) is the left [resp., right] reflection of \( I_j \) in \( s_j \) for \( j = 1, 2, \ldots, k - 1 \). Analogously we define what is meant by a point \( p' \) being a finite left or right reflection of a point \( p \) in \( S \).
Lemma 1. If $S \subseteq \mathbb{R}$ and $l(0, r, S)/r < 1/2$ for $0 < r < \delta$, then each open interval $(a, b)$ contained in $(0, \delta)$ [resp., $(-\delta, 0)$] has a finite left [resp., right] reflection in $S$ that contains 0.

Proof. Suppose $I \equiv (a, b) \subseteq (0, \delta)$. If $a = 0$, then the reflection of $I$ in any point of $S \cap (0, b/2)$ contains 0. Suppose $a \neq 0$. It follows from the hypothesis that there exists a point $s$ in $S \cap (a/2, a)$, and clearly the reflection $(a', b')$ of $(a, b)$ in $s$ is such that $b' > 0$. If $0 \not\in [a', b')$, then reflect $(a', b')$ in some point $s'$ in $S \cap (a'/2, a')$ to obtain an interval $(a'', b'')$ with $b'' > 0$. Since each of the reflected intervals has the same length as $I$, a finite repetition of this process will produce a finite left reflection $(a^*, b^*)$ of $I$ in $S$ with $0 \in [a^*, b^*)$ and $b^* > 0$. Either $(a^*, b^*)$ contains $0$ or $a^* = 0$ and the reflection of $(a^*, b^*)$ in any point of $S \cap (0, b^*/2)$ contains 0. Thus the lemma is established in the case $I \subseteq (0, \delta)$, and the case $I \subseteq (-\delta, 0)$ is handled similarly.

Lemma 2. If $S \subseteq \mathbb{R}$ and $l(0, r, S)/r < \lambda < 1/2$ for $0 < r < \delta$, then, for every $\varepsilon > 0$, each point $p$ in $(0, \delta)$ [resp., $(-\delta, 0)$] has a finite left [resp., right] reflection in $S$ that is contained in $(0, \varepsilon)$ [resp., $(-\varepsilon, 0)$].

Proof. Suppose $p \in (0, \delta)$. It follows from the hypothesis that there exists a point $s$ in $S \cap (p/2, p/2 + \lambda p)$. If $p_1$ is the reflection of $p$ in $s$, then $0 < p_1 = 2s - p < 2\lambda p$. Similarly we can reflect $p_1$ in some point $s_1$ in $S \cap (p_1/2, p_1/2 + \lambda p_1)$ to obtain a point $p_2$ with $0 < p_2 < 2\lambda p_1 < (2\lambda)^2 p$.

Continuing this reflection process inductively, we obtain a sequence $p_1, p_2, \ldots$ of points with $0 < p_n < (2\lambda)^np$ for each index $n$. Then, since $0 < \lambda < 1/2$, it follows that $p_n \downarrow 0$. This establishes the lemma for $p \in (0, \delta)$, and the proof for $p \in (-\delta, 0)$ is similar.

3. Porosity lemmas applied to the sets $S_n$. In the next section, we will prove that the set $\mathcal{F} - \mathcal{E}$ is a $\sigma$-porous set for functions $f \in \mathcal{F}$, and we will find it convenient to use the decomposition $\mathcal{F} = \bigcup_{n=1}^{\infty} S_n$ where

$$S_n = \{ x : f(x - h) < f(x + h) \text{ for } 0 < h < 1/n \}.$$  

In that proof, we make use of certain properties that an arbitrary function $f : \mathbb{R} \to \mathbb{R}$ possesses at a point where the porosity of $S_n$ is $< 1/2$. These properties are given in the next two lemmas.

Lemma 3. Let $f : \mathbb{R} \to \mathbb{R}$ be arbitrary, and let $\mathcal{C}(f)$ denote the set of points where $f$ is continuous. If $l(0, r, S_n)/r < 1/2$ for $0 < r < \delta < 1/n$, then

$$\sup_{-\delta < x < 0} f(x) \leq \lim_{x \to 0} f(x) \leq \lim_{x \to 0} f(0) \leq \inf_{0 < x < \delta} f(x) \leq \inf_{x \in \mathcal{C}(f)} f(x).$$ (*)
Proof. By hypothesis, there exists a strictly decreasing sequence \( s_1, s_2, \ldots \) of points in \( S_n \cap (0, \delta) \) that converges to 0. Since \( s_k \in S_n \), we have \( f(0) < f(2s_k) \) which yields the third inequality in (\(*\)). A similar proof establishes the second inequality in (\(*\)).

Now choose a point \( c \in \mathcal{C}(f) \cap (0, \delta) \) and let \( \varepsilon > 0 \) be given. By the continuity of \( f \) at \( c \), there is an open subinterval \( I \) of \( (0, \delta) \) such that \( f(x) < f(c) + \varepsilon \) for each \( x \in I \). According to Lemma 1, there is a finite left reflection \( I' \) of \( I \) in \( S_n \) with \( 0 \in I' \). Since \( I' \) is a left reflection of \( I \) in \( S_n \), we have \( f(x) < f(c) + \varepsilon \) for each \( x \in I' \). The last inequality in (\(*\)) now follows readily, and the first inequality in (\(*\)) is established in a similar manner. This completes the proof of the lemma.

**Lemma 4.** Let \( f: \mathbb{R} \to \mathbb{R} \) be arbitrary. If \( l(0, r, S_n)/r < \lambda < 1/2 \) for \( 0 < r < \delta < \sqrt{n} \), then
\[
\lim_{x \to 0^+} f(x) = \inf_{0 < x < \delta} f(x) \quad \text{and} \quad \lim_{x \to 0^-} f(x) = \sup_{-\delta < x < 0} f(x). \quad (\dagger)
\]

Proof. Set \( \alpha = \inf_{0 < x < \delta} f(x) \) and let \( \varepsilon > 0 \) be given. Choose \( x \in (0, \delta) \) for which \( f(x) < \alpha + \varepsilon \). By Lemma 2 there exists a sequence \( x = x_1, x_2, \ldots \) of points such that \( x_j \to 0 \) and \( x_{j+1} \) is a left reflection of \( x_j \) in \( S_n \) for each index \( j \). Hence \( f(x_{j+1}) < f(x_j) \) for each index \( j \) and it follows that \( \lim_{j \to \infty} f(x_j) < \alpha + \varepsilon \). This proves the first equality in (\(\dagger\)); the proof of the second equality is similar, and the lemma is established.

**4. The set \( \mathcal{S} \subseteq \mathcal{G}_\ast \).**

**Theorem 1.** If \( f \in \mathcal{G}_\ast \), then \( \mathcal{S} = \mathcal{G}_\ast \) is \( \sigma \)-porous; indeed, it is a countable union of sets, each of which has porosity \( \geq 1/2 \) at each of its points.

Proof. Let \( H_n \) denote the set of all points in \( S_n \) at which \( S_n \) has porosity \( < 1/2 \), and choose a point \( \hat{x} \in H_n \). Let \( \lambda \) and \( \delta \) be numbers such that \( l(\hat{x}, r, S_n)/r < \lambda < 1/2 \) for \( 0 < r < \delta < 1/n \). Replacing 0 by \( \hat{x} \) in Lemma 3, we have
\[
\sup_{0 < h < \delta} f(\hat{x} - h) < \inf_{0 < h < \delta} f(\hat{x} + h). \quad (\ast)
\]
However, if equality holds in (\(\ast\)), it follows from Lemma 3 that \( f(x) \) is continuous at \( \hat{x} \). So, replacing 0 by \( \hat{x} \) in Lemma 4, we have that
\[
\sup_{0 < h < \delta} f(\hat{x} - h) = f(\hat{x}) = \inf_{0 < h < \delta} f(\hat{x} + h),
\]
that is, \( \hat{x} \in \mathcal{G}_\ast \). But this contradicts \( \hat{x} \in H_n \) and so the strict inequality must hold in (\(\ast\)).

Now for each rational number \( \alpha \) and each positive integer \( k \) let \( A_{\alpha k} \) denote the set of all points \( x \in \mathbb{R} \) for which
\[
\sup_{0 < h < 1/k} f(x - h) < \alpha < \inf_{0 < h < 1/k} f(x + h).
\]

From the discussion in the previous paragraph, it follows that \( H_n \) is a subset of the countable union of all the sets \( A_{ak} \); furthermore, since \( C(f) \) is dense in \( R \), it is easily observed that each of the sets \( A_{ak} \) is an isolated set, more specifically, if \( x \in A_{ak} \) then \( A_{ak} \cap (x - 2/k, x + 2/k) = \{x\} \). Hence \( H_n \) is a countable set, and we can write

\[
S_n - g = [(S_n - g) - H_n] \cup \left( \bigcup_{x \in H_n} \{x\} \right),
\]

where each set in the union on the right has porosity \( \geq 1/2 \) at each of its points. This proves the theorem.

5. Dini derivâtes and symmetric derivâtes. The upper right and the upper left Dini derivâtes of the function \( f: R \to R \) are respectively

\[
D^+ f(x) \equiv \lim_{h \to 0^+} \sup \frac{f(x + h) - f(x)}{h}
\]

and

\[
D^- f(x) \equiv \lim_{h \to 0^-} \sup \frac{f(x + h) - f(x)}{h}.
\]

The lower Dini derivâtes \( D_- f(x) \) and \( D_+ f(x) \) are the corresponding \( \lim \inf \)s.

**Theorem 2.** If \( f \in \mathcal{G} \), then for all but a \( \sigma \)-porous set of points both of the following equalities hold:

(i) \( D_f f(x) = \min\{D_+ f(x), D_- f(x)\} \),

(ii) \( D^f f(x) = \max\{D^+ f(x), D^- f(x)\} \).

**Proof.** We first observe that for each point \( x \) we have

\[
\min\{D_- f(x), D_+ f(x)\} \leq D_f f(x) \leq D^f f(x) \leq \max\{D^- f(x), D^+ f(x)\}.
\]

\((*)\)

Now for each rational number \( \alpha \) set

\[
N(f, \alpha) = \{x: D_f f(x) > \alpha > \min\{D_- f(x), D_+ f(x)\}\}.
\]

Then, in light of the first inequality in \((*)\), we see that in order to prove the statement of the theorem concerning equality (i), it suffices to show that each set \( N(f, \alpha) \) is \( \sigma \)-porous. Furthermore, it is sufficient to show that \( N(f, 0) \) is \( \sigma \)-porous because \( N(f, \alpha) = N(g, 0) \) for the function \( g(x) = f(x) - \alpha x \). But this is a direct consequence of Theorem 1 since \( N(f, 0) \subset \mathcal{G} - \mathcal{G} \).

Now, replacing \( f(x) \) with \(-f(x)\), we use this same argument to establish the statement of the theorem concerning equality (ii); hence, the proof is complete.

6. Consequences of Theorem 2. An immediate consequence of Theorem 2 is the following theorem which says that, for functions in class \( \mathcal{G} \), the ordinary
derivative exists at most points where the symmetric derivative exists.

**Theorem 3.** If $f \in \mathcal{F}$, then $f'(x)$ exists at all but a $\sigma$-porous set of points where $f^*(x)$ exists.

Now, according to the theorem of Khintchine given in §1 of this paper, a measurable symmetrically differentiable function must be differentiable in the ordinary sense almost everywhere and is therefore in class $\mathcal{F}$. This observation together with Theorem 3 yields the following result which was announced in the abstract of this article.

**Theorem 4.** If $f$ is measurable and symmetrically differentiable on $\mathbb{R}$, then $f'(x)$ exists for all but a $\sigma$-porous set of points.

Although it has been known for some time that a continuous symmetrically differentiable function is differentiable in the ordinary sense at each point of $\mathbb{R}$ except possibly for a set of the first category and measure zero, it was only recently shown that this exceptional set need not be countable. That is, based upon a construction of J. Foran [2], we see that this exceptional set can equal certain perfect sets of measure zero. In this connection, Foran [2] posed the question as to whether each perfect set of measure zero could be such an exceptional set. Theorem 4 provides a negative answer to this question; that is, the non-$\sigma$-porous perfect set of measure zero constructed by L. Zajiček [5] cannot be such an exceptional set. Nevertheless, the following specialization of Foran’s question remains open:

Is each perfect $\sigma$-porous set precisely the set of points where some continuous symmetrically differentiable function fails to be differentiable in the ordinary sense?

The final consequence of Theorem 2 that we wish to mention concerns points of density of a measurable set. It is well known that the set of points $x$ where the symmetric metric density of a measurable set $M$ exists but the ordinary metric density does not constitutes a set of the first category and measure zero. By considering $f(x)$ to be an integral of the characteristic function of $M$, we see that Theorem 3 allows us to say more.

**Theorem 5.** If $M$ is a measurable subset of $\mathbb{R}$, then for all but a $\sigma$-porous set of points $x \in \mathbb{R}$ the existence of the symmetric metric density of $M$ at $x$ implies the existence of the ordinary metric density of $M$ at $x$.

**References**

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