

## SYMMETRIC AND ORDINARY DIFFERENTIATION

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**ABSTRACT.** In 1927, A. Khintchine proved that a measurable symmetrically differentiable function  $f$  mapping the real line  $R$  into itself is differentiable in the ordinary sense at each point of  $R$  except possibly for a set of Lebesgue measure zero. Here it is shown that this exceptional set is also of the first Baire category; even more, it is shown to be a  $\sigma$ -porous set of E. P. Dolzhenko.

**1. Introduction.** Let  $f$  be a real valued function defined on the real line  $R$ . The upper symmetric derivate of  $f$  at  $x \in R$  is

$$D^s f(x) \equiv \limsup_{h \rightarrow 0} \frac{f(x+h) - f(x-h)}{2h}.$$

The lower symmetric derivate  $D_s f(x)$  of  $f$  at  $x$  is the corresponding limit inferior. If  $D^s f(x) = D_s f(x)$ , then this common value is called the symmetric derivative of  $f$  at  $x$  and is denoted  $f^s(x)$ .

It is readily observed that the existence of the ordinary derivative  $f'(x)$  implies the existence of  $f^s(x)$ , with the two values being equal. Concerning the reverse implication, Khintchine [4] proved: *a measurable function  $f: R \rightarrow R$  has a finite derivative  $f'(x)$  at almost every point where  $D^s f(x) < \infty$ .* We observe that the exceptional set in this theorem, although of Lebesgue measure zero, need not be of the first Baire category even if the condition  $D^s f(x) < \infty$  is replaced by the stronger condition that  $f^s(x)$  exists and is finite. To see this, consider  $f$  to be the characteristic function of an additive group  $G$  of real numbers that is of measure zero and of the second category. Then  $f^s(x) = 0$  for each  $x \in G$ , but  $f'(x)$  exists at no point of  $G$ . We also observe that the exceptional set in Khintchine's theorem need not be of the first category when strong continuity conditions are placed on the function. Indeed, let  $Z$  be any bounded subset of  $R$  that is both of measure 0 and of the second category. According to C. Goffman [3], there exists a bounded measurable subset  $S$  of  $R$  whose metric density does not exist at any point of  $Z$ . Let  $\chi_S$  be the characteristic function of  $S$  and set

$$f(x) = \int_{-\infty}^x \chi_S(t) dt.$$

Then  $f$  is a Lipschitz function (with its symmetric derivates bounded in

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absolute value by 1 everywhere), but  $f'(x)$  does not exist at any point of  $Z$ .

With the two examples of the previous paragraph in mind, we conclude that in order to obtain a category analogue of Khintchine's theorem we must simultaneously impose some sort of continuity conditions on  $f$  and replace the condition  $D^2f(x) < \infty$  by the stronger condition that  $f^s(x)$  exists. In so doing, our result (Theorem 3 in §6) says that if  $f$  belongs to the function class

$$\mathcal{F} \equiv \{f: \text{the points of continuity of } f \text{ are dense in } R\},$$

then the existence of  $f^s(x)$  implies the existence of  $f'(x)$  at all but a  $\sigma$ -porous set of points (see §2 for the definition of these sets).

This theorem is a direct consequence of Theorem 2 in §5, which gives a relationship between the symmetric derivatives and the Dini derivatives of functions  $f$  in  $\mathcal{F}$ . The proof of Theorem 2 rests on the determination of the size of the difference set  $\mathcal{G}^* - \mathcal{G}$ , where

$$\mathcal{G}^* = \{x: f(x-h) \leq f(x+h) \text{ for sufficiently small } h > 0\}$$

and

$$\mathcal{G} = \{x: f(x-h) \leq f(x) \leq f(x+h) \text{ for sufficiently small } h > 0\}.$$

This determination is made in §4, while §§2 and 3 are devoted to the establishment of some preliminary results that will be used there.

**2. Porosity lemmas.** The notion of porosity is due to E. P. Dolženko [1]. The *porosity of the set*  $S \subset R$  *at the point*  $x \in R$  *is defined to be the nonnegative value*

$$\limsup_{r \downarrow 0} \frac{l(x, r, S)}{r},$$

where  $l(x, r, S)$  denotes the length of the largest open interval contained in the set  $(x-r, x+r) \cap (R-S)$ . The set  $S$  is called *porous* if it has positive porosity at each of its points, and it is called  *$\sigma$ -porous* if it is a countable union of porous sets. It follows readily from the definition that a  $\sigma$ -porous set is both of the first category and of measure zero. On the other hand, L. Zajíček [5] has constructed a perfect set of measure zero that is not  $\sigma$ -porous.

We now introduce some terminology that will be used in the statements of the following two lemmas concerning points at which a set has porosity  $< 1/2$ : Let  $I$  be an open interval and let  $s$  be a point of  $R$ . The *reflection of*  $I$  *in*  $s$  *is defined to be the open interval*  $\{2s-x: x \in I\}$ ; it is called a *left* [resp., *right*] *reflection of*  $I$  *in*  $s$  *if*  $s \leq (a+b)/2$  [resp.,  $s \geq (a+b)/2$ ], where  $I \equiv (a, b)$ . If  $S \subset R$ , then an open interval  $I'$  is said to be a *finite left* [resp., *right*] *reflection of*  $I$  *in*  $S$  *if there exists a finite collection*  $\{I_1, I_2, \dots, I_k\}$  *of open intervals and a corresponding subset*  $\{s_1, s_2, \dots, s_{k-1}\}$  *of*  $S$  *such that*  $I = I_1$ ,  $I' = I_k$ , *and*  $I_{j+1}$  *is the left* [resp., *right*] *reflection of*  $I_j$  *in*  $s_j$  *for*  $j = 1, 2, \dots, k-1$ . Analogously we define what is meant by a point  $p'$  being a finite left or right reflection of a point  $p$  in  $S$ .

LEMMA 1. *If  $S \subset R$  and  $l(0, r, S)/r < 1/2$  for  $0 < r < \delta$ , then each open interval  $(a, b)$  contained in  $(0, \delta)$  [resp.,  $(-\delta, 0)$ ] has a finite left [resp., right] reflection in  $S$  that contains 0.*

PROOF. Suppose  $I \equiv (a, b) \subset (0, \delta)$ . If  $a = 0$ , then the reflection of  $I$  in any point of  $S \cap (0, b/2)$  contains 0. Suppose  $a \neq 0$ . It follows from the hypothesis that there exists a point  $s$  in  $S \cap (a/2, a)$ , and clearly the reflection  $(a', b')$  of  $(a, b)$  in  $s$  is such that  $b' > 0$ . If  $0 \notin [a', b')$ , then reflect  $(a', b')$  in some point  $s'$  in  $S \cap (a'/2, a')$  to obtain an interval  $(a'', b'')$  with  $b'' > 0$ . Since each of the reflected intervals has the same length as  $I$ , a finite repetition of this process will produce a finite left reflection  $(a^*, b^*)$  of  $I$  in  $S$  with  $0 \in [a^*, b^*)$  and  $b^* > 0$ . Either  $(a^*, b^*)$  contains 0 or  $a^* = 0$  and the reflection of  $(a^*, b^*)$  in any point of  $S \cap (0, b^*/2)$  contains 0. Thus the lemma is established in the case  $I \subset (0, \delta)$ , and the case  $I \subset (-\delta, 0)$  is handled similarly.

LEMMA 2. *If  $S \subset R$  and  $l(0, r, S)/r < \lambda < 1/2$  for  $0 < r < \delta$ , then, for every  $\epsilon > 0$ , each point  $p$  in  $(0, \delta)$  [resp.,  $(-\delta, 0)$ ] has a finite left [resp., right] reflection in  $S$  that is contained in  $(0, \epsilon)$  [resp.,  $(-\epsilon, 0)$ ].*

PROOF. Suppose  $p \in (0, \delta)$ . It follows from the hypothesis that there exists a point  $s$  in  $S \cap (p/2, p/2 + \lambda p)$ . If  $p_1$  is the reflection of  $p$  in  $s$ , then

$$0 < p_1 = 2s - p < 2\lambda p.$$

Similarly we can reflect  $p_1$  in some point  $s_1$  in  $S \cap (p_1/2, p_1/2 + \lambda p_1)$  to obtain a point  $p_2$  with

$$0 < p_2 < 2\lambda p_1 < (2\lambda)^2 p.$$

Continuing this reflection process inductively, we obtain a sequence  $p_1, p_2, \dots$  of points with  $0 < p_n < (2\lambda)^n p$  for each index  $n$ . Then, since  $0 < \lambda < 1/2$ , it follows that  $p_n \downarrow 0$ . This establishes the lemma for  $p \in (0, \delta)$ , and the proof for  $p \in (-\delta, 0)$  is similar.

**3. Porosity lemmas applied to the sets  $S_n$ .** In the next section, we will prove that the set  $\mathcal{G}^* - \mathcal{G}$  is a  $\sigma$ -porous set for functions  $f \in \mathcal{F}$ , and we will find it convenient to use the decomposition  $\mathcal{G}^* = \cup_{n=1}^\infty S_n$  where

$$S_n = \{x: f(x - h) \leq f(x + h) \text{ for } 0 < h < 1/n\}.$$

In that proof, we make use of certain properties that an arbitrary function  $f: R \rightarrow R$  possesses at a point where the porosity of  $S_n$  is  $< 1/2$ . These properties are given in the next two lemmas.

LEMMA 3. *Let  $f: R \rightarrow R$  be arbitrary, and let  $\mathcal{C}(f)$  denote the set of points where  $f$  is continuous. If  $l(0, r, S_n)/r < 1/2$  for  $0 < r < \delta < 1/n$ , then*

$$\sup_{\substack{-\delta < x < 0 \\ x \in \mathcal{C}(f)}} f(x) \leq \lim_{x \rightarrow 0} f(x) \leq f(0) \leq \overline{\lim}_{x \rightarrow 0} f(x) \leq \inf_{\substack{0 < x < \delta \\ x \in \mathcal{C}(f)}} f(x). \quad (*)$$

PROOF. By hypothesis, there exists a strictly decreasing sequence  $s_1, s_2, \dots$  of points in  $S_n \cap (0, \delta)$  that converges to 0. Since  $s_k \in S_n$ , we have  $f(0) \leq f(2s_k)$  which yields the third inequality in (\*). A similar proof establishes the second inequality in (\*).

Now choose a point  $c \in \mathcal{C}(f) \cap (0, \delta)$  and let  $\epsilon > 0$  be given. By the continuity of  $f$  at  $c$ , there is an open subinterval  $I$  of  $(0, \delta)$  such that  $f(x) < f(c) + \epsilon$  for each  $x \in I$ . According to Lemma 1, there is a finite left reflection  $I'$  of  $I$  in  $S_n$  with  $0 \in I'$ . Since  $I'$  is a left reflection of  $I$  in  $S_n$ , we have  $f(x) < f(c) + \epsilon$  for each  $x \in I'$ . The last inequality in (\*) now follows readily, and the first inequality in (\*) is established in a similar manner. This completes the proof of the lemma.

LEMMA 4. Let  $f: R \rightarrow R$  be arbitrary. If  $l(0, r, S_n)/r < \lambda < 1/2$  for  $0 < r < \delta < 1/n$ , then

$$\lim_{x \rightarrow 0^+} f(x) = \inf_{0 < x < \delta} f(x) \quad \text{and} \quad \overline{\lim}_{x \rightarrow 0^-} f(x) = \sup_{-\delta < x < 0} f(x). \quad (\dagger)$$

PROOF. Set  $\alpha = \inf_{0 < x < \delta} f(x)$  and let  $\epsilon > 0$  be given. Choose  $x \in (0, \delta)$  for which  $f(x) < \alpha + \epsilon$ . By Lemma 2 there exists a sequence  $x = x_1, x_2, \dots$  of points such that  $x_j \rightarrow 0$  and  $x_{j+1}$  is a left reflection of  $x_j$  in  $S_n$  for each index  $j$ . Hence  $f(x_{j+1}) \leq f(x_j)$  for each index  $j$  and it follows that  $\lim_{j \rightarrow \infty} f(x_j) < \alpha + \epsilon$ . This proves the first equality in ( $\dagger$ ); the proof of the second equality is similar, and the lemma is established.

4. The set  $\mathcal{G}^* - \mathcal{G}$ .

THEOREM 1. If  $f \in \mathcal{F}$ , then  $\mathcal{G}^* - \mathcal{G}$  is  $\sigma$ -porous; indeed, it is a countable union of sets, each of which has porosity  $\geq 1/2$  at each of its points.

PROOF. Let  $H_n$  denote the set of all points in  $S_n - \mathcal{G}$  at which  $S_n$  has porosity  $< 1/2$ , and choose a point  $\hat{x} \in H_n$ . Let  $\lambda$  and  $\delta$  be numbers such that  $l(\hat{x}, r, S_n)/r < \lambda < 1/2$  for  $0 < r < \delta < 1/n$ . Replacing 0 by  $\hat{x}$  in Lemma 3, we have

$$\sup_{\substack{0 < h < \delta \\ \hat{x} - h \in \mathcal{C}(f)}} f(\hat{x} - h) \leq \inf_{\substack{0 < h < \delta \\ \hat{x} + h \in \mathcal{C}(f)}} f(\hat{x} + h). \quad (*)$$

However, if equality holds in (\*), it follows from Lemma 3 that  $f(x)$  is continuous at  $\hat{x}$ . So, replacing 0 by  $\hat{x}$  in Lemma 4, we have that

$$\sup_{0 < h < \delta} f(\hat{x} - h) = f(\hat{x}) = \inf_{0 < h < \delta} f(\hat{x} + h),$$

that is,  $\hat{x} \in \mathcal{G}$ . But this contradicts  $\hat{x} \in H_n$  and so the strict inequality must hold in (\*).

Now for each rational number  $\alpha$  and each positive integer  $k$  let  $A_{\alpha k}$  denote the set of all points  $x \in R$  for which

$$\sup_{\substack{0 < h < 1/k \\ x - h \in \mathcal{C}(f)}} f(x - h) < \alpha < \inf_{\substack{0 < h < 1/k \\ x + h \in \mathcal{C}(f)}} f(x + h).$$

From the discussion in the previous paragraph, it follows that  $H_n$  is a subset of the countable union of all the sets  $A_{\alpha k}$ ; furthermore, since  $\mathcal{C}(f)$  is dense in  $R$ , it is easily observed that each of the sets  $A_{\alpha k}$  is an isolated set, more specifically, if  $x \in A_{\alpha k}$  then  $A_{\alpha k} \cap (x - 2/k, x + 2/k) = \{x\}$ . Hence  $H_n$  is a countable set, and we can write

$$S_n - \mathcal{G} = [(S_n - \mathcal{G}) - H_n] \cup \left( \bigcup_{x \in H_n} \{x\} \right),$$

where each set in the union on the right has porosity  $\geq 1/2$  at each of its points. This proves the theorem.

**5. Dini derivates and symmetric derivates.** The *upper right* and the *upper left Dini derivates* of the function  $f: R \rightarrow R$  are respectively

$$D^+ f(x) \equiv \limsup_{h \rightarrow 0^+} \frac{f(x + h) - f(x)}{h}$$

and

$$D^- f(x) \equiv \limsup_{h \rightarrow 0^-} \frac{f(x + h) - f(x)}{h}.$$

The *lower Dini derivates*  $D_+ f(x)$  and  $D_- f(x)$  are the corresponding *lim inf*'s.

**THEOREM 2.** *If  $f \in \mathcal{F}$ , then for all but a  $\sigma$ -porous set of points both of the following equalities hold:*

- (i)  $D_s f(x) = \min\{D_+ f(x), D_- f(x)\}$ ,
- (ii)  $D^s f(x) = \max\{D^+ f(x), D^- f(x)\}$ .

**PROOF.** We first observe that for each point  $x$  we have

$$\min\{D_- f(x), D_+ f(x)\} \leq D_s f(x) \leq D^s f(x) \leq \max\{D^- f(x), D^+ f(x)\}. \tag{*}$$

Now for each rational number  $\alpha$  set

$$N(f, \alpha) = \{x: D_s f(x) > \alpha > \min\{D_- f(x), D_+ f(x)\}\}.$$

Then, in light of the first inequality in (\*), we see that in order to prove the statement of the theorem concerning equality (i), it suffices to show that each set  $N(f, \alpha)$  is  $\sigma$ -porous. Furthermore, it is sufficient to show that  $N(f, 0)$  is  $\sigma$ -porous because  $N(f, \alpha) = N(g, 0)$  for the function  $g(x) = f(x) - \alpha x$ . But this is a direct consequence of Theorem 1 since  $N(f, 0) \subset \mathcal{G} - \mathcal{G}$ .

Now, replacing  $f(x)$  with  $-f(x)$ , we use this same argument to establish the statement of the theorem concerning equality (ii); hence, the proof is complete.

**6. Consequences of Theorem 2.** An immediate consequence of Theorem 2 is the following theorem which says that, for functions in class  $\mathcal{F}$ , the ordinary

derivative exists at *most* points where the symmetric derivative exists.

**THEOREM 3.** *If  $f \in \mathcal{F}$ , then  $f'(x)$  exists at all but a  $\sigma$ -porous set of points where  $f^s(x)$  exists.*

Now, according to the theorem of Khintchine given in §1 of this paper, a measurable symmetrically differentiable function must be differentiable in the ordinary sense almost everywhere and is therefore in class  $\mathcal{F}$ . This observation together with Theorem 3 yields the following result which was announced in the abstract of this article.

**THEOREM 4.** *If  $f$  is measurable and symmetrically differentiable on  $R$ , then  $f'(x)$  exists for all but a  $\sigma$ -porous set of points.*

Although it has been known for some time that a continuous symmetrically differentiable function is differentiable in the ordinary sense at each point of  $R$  except possibly for a set of the first category and measure zero, it was only recently shown that this exceptional set need not be countable. That is, based upon a construction of J. Foran [2], we see that this exceptional set can equal certain perfect sets of measure zero. In this connection, Foran [2] posed the question as to whether each perfect set of measure zero could be such an exceptional set. Theorem 4 provides a negative answer to this question; that is, the non- $\sigma$ -porous perfect set of measure zero constructed by L. Zajíček [5] cannot be such an exceptional set. Nevertheless, the following specialization of Foran's question remains open:

*Is each perfect  $\sigma$ -porous set precisely the set of points where some continuous symmetrically differentiable function fails to be differentiable in the ordinary sense?*

The final consequence of Theorem 2 that we wish to mention concerns points of density of a measurable set. It is well known that the set of points  $x$  where the symmetric metric density of a measurable set  $M$  exists but the ordinary metric density does not constitutes a set of the first category and measure zero. By considering  $f(x)$  to be an integral of the characteristic function of  $M$ , we see that Theorem 3 allows us to say more.

**THEOREM 5.** *If  $M$  is a measurable subset of  $R$ , then for all but a  $\sigma$ -porous set of points  $x \in R$  the existence of the symmetric metric density of  $M$  at  $x$  implies the existence of the ordinary metric density of  $M$  at  $x$ .*

#### REFERENCES

1. E. P. Dolženko, *Boundary properties of arbitrary functions*, Math. USSR-Izv. **1** (1967), 1–12.
2. J. Foran, *The symmetric and ordinary derivative*, Real Analysis Exchange **2** (1977), 105–108.
3. C. Goffman, *On Lebesgue's density theorem*, Proc. Amer. Math. Soc. **1** (1950), 384–388.
4. A. Khintchine, *Recherches sur la structure des fonctions mesurables*, Fund. Math. **9** (1927), 212–279.
5. L. Zajíček, *Sets of  $\sigma$ -porosity and sets of  $\sigma$ -porosity ( $q$ )*, Časopis Pěst. Mat. **101** (1976), 350–359.

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