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LERAY-SCHAUDER PRINCIPLES FOR CONDENSING MULTI-VALUED MAPPINGS IN TOPOLOGICAL LINEAR SPACES

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ABSTRACT. By establishing the existence of an equivalent fixed point problem it is shown without any recourse to degree theory or to the theory of homotopy-extension-theorems that all fixed point theorems of Leray-Schauder type for condensing (single- or multi-valued) mappings in topological linear spaces can immediately be deduced from the corresponding fixed point theorems of Schauder type.

Throughout our discussion K will denote a closed convex subset of a topological linear space E such that the origin 0 of E belongs to K and for $U \subset K$ we use $\mathrm{cl}_K(U)$ to denote the closure of U (in K) and $\partial_K U$ to denote the boundary of U (in K).

If Z is a topological space and T: $Z \to 2^K_1$, then T is said to be *upper semicontinuous* (u.s.c.) if and only if $T^{-1}(A)$ is a closed set for all subsets A of K, which are closed (in K).

If C is a lattice with a minimal element, which we will denote by 0, too, then a map χ: $2^K \to C$ is called a *measure of noncompactness of* K provided that the following conditions hold for any X, Y in 2^K:

(1) $\chi(X) = 0$ if and only if $\overline{X}$ is relatively compact,

(2) $\chi(\overline{\mathrm{co}}(X)) = \chi(X)$, where $\overline{\mathrm{co}}(X)$ denotes the convex closure of X,

(3) $\chi(X \cup Y) = \max\{\chi(X), \chi(Y)\}$.

If χ is a measure of noncompactness of K, $D \subset K$ and f: $D \to 2^K$, then f is called χ-*condensing* provided that if $X \subset D$ and $\chi(X) \leqslant \chi(f(X))$, then X is relatively compact, i.e., $\chi(X) = 0$.

Our main result is the following theorem.

THEOREM. *Let* χ *be a measure of noncompactness of* K, $\mathfrak{M}$ *be a set of nonempty compact subsets of* K, $U \subset K$ *be an open neighborhood of* 0 (*in* K) *and* H: $[0, 1] \times \mathrm{cl}_K(U) \to \mathfrak{M}$ *be u.s.c. Suppose that*

(4) $x \notin H(t, x)$ *for* $t \in [0, 1]$ *and* $x \in \partial_K U$,

(5) $X \subset \mathrm{cl}_K(U)$ *and* $\chi(X) \leqslant \chi(H([0, 1] \times X))$ *imply that* X *is relatively compact*,

(6) $\mu x \notin H(1, x)$ *for* $\mu > 1$ *and* $x \in \partial_K U$,

and

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(7) *$tM \in \mathfrak{M}$ for $t \in [0, 1]$ and $M \in \mathfrak{M}$.*
Then there exists an u.s.c. χ-condensing mapping g: $K \to \mathfrak{M}$ such that for all $x \in K$: $x \in g(x)$ if and only if $x \in \mathrm{cl}_K(U)$ and $x \in H(0, x)$.

PROOF. Let $R \subset K$ be defined by

$$R := \{x \in \mathrm{cl}_K(U): x \in H(t, x) \text{ for some } t \in [0, 1]\}$$
$$\cup \{x \in \mathrm{cl}_K(U): x \in tH(1, x) \text{ for some } t \in [0, 1]\}.$$

Then R is nonempty ($0 \in R$) and compact (H is u.s.c. and $R \subset \overline{\mathrm{co}}(\{0\} \cup H([0, 1] \times R))$) such that $R \cap \partial_K U = \varnothing$ (by (4) and (6)). Since K is a completely regular topological space there exists therefore a continuous mapping λ: $K \to [0, 1]$ such that $\lambda(R) \subset \{0\}$ and $\lambda(\partial_K U) \subset \{1\}$. Let now g: $K \to \mathfrak{M}$ be defined by

$$g(x) := \begin{cases} H(2\lambda(x), x), & \lambda(x) \leq \frac{1}{2} \text{ and } x \in \mathrm{cl}_K(U), \\ 2(1 - \lambda(x))H(1, x), & \lambda(x) \geq \frac{1}{2} \text{ and } x \in \mathrm{cl}_K(U), \\ \{0\}, & x \notin \mathrm{cl}_K(U). \end{cases}$$

Then

(8) g is u.s.c.,

(9) g is χ-condensing.

(10) For all $x \in K$: $x \in g(x)$ if and only if $x \in \mathrm{cl}_K(U)$ and $x \in H(0, x)$.

PROOF OF (8). H is u.s.c., λ is continuous and $\lambda(\partial_K U) \subset \{1\}$.

PROOF OF (9). Let $X \subset K$ such that $\chi(X) \leq \chi(g(X))$. By definition of g, we have $g(X) \subset \overline{\mathrm{co}}(\{0\} \cup H([0, 1] \times (\mathrm{cl}_K(U) \cap X)))$. If $\mathrm{cl}_K(U) \cap X = \varnothing$, then $\chi(X) \leq \chi(g(X)) \leq \chi(\{0\}) = 0$ and therefore X is relatively compact. If $\mathrm{cl}_K(U) \cap X \neq \varnothing$, then (2) and (3) imply

$$\begin{aligned} \chi(\mathrm{cl}_K(U) \cap X) &\leq \chi(X) \leq \chi(g(X)) \\ &\leq \chi\left(\overline{\mathrm{co}}\left(\{0\} \cup H\left([0, 1] \times (\mathrm{cl}_K(U) \cap X)\right)\right)\right) \\ &\leq \chi\left(H\left([0, 1] \times (\mathrm{cl}_K(U) \cap X)\right)\right) \end{aligned}$$

so that $\mathrm{cl}_K(U) \cap X$ is relatively compact by (5). But then $H([0, 1] \times (\mathrm{cl}_K(U) \cap X))$ is relatively compact (H is u.s.c. and $H(t, x)$ is compact for $t \in [0, 1]$ and $x \in \mathrm{cl}_K(U)$) and therefore by (1): $\chi(X) \leq \chi(H([0, 1] \times (\mathrm{cl}_K(U) \cap X))) = 0$, i.e., X is relatively compact.

PROOF OF (10). Let $x \in K$. If $x \in g(x)$, then $x \in R$, hence $\lambda(x) = 0$ and therefore $x \in \mathrm{cl}_K(U)$ and $x \in H(0, x)$. Conversely, if $x \in \mathrm{cl}_K(U)$ and $x \in H(0, x)$, then $x \in R$, hence $\lambda(x) = 0$ and therefore $x \in g(x)$.

REMARK 1. Variations of this proof have been given by several authors to establish some more or less related results (see e.g. Browder **[1]**, Edmunds-Webb **[2]**, Fitzpatrick-Petryshyn **[4]**, Granas **[5]**, Hahn **[6]**, Potter **[9]** and Webb **[10]**).

REMARK 2. If in addition to the hypotheses of our Theorem the sets K and

$\mathfrak{M}$ satisfy:

(11) For all u.s.c. χ-condensing mappings $f: K \to \mathfrak{M}$ there is $x \in K$ such that $x \in f(x)$,

then there is $x \in \mathrm{cl}_K(U)$ such that $x \in H(0, x)$.

This immediate consequence of our Theorem shows, that in order to establish a Leray-Schauder principle for condensing (single- or multi-valued) mappings one has to verify only (11) for the sets K and $\mathfrak{M}$.

Using this remark, it is clear that the following very general result is only a simple consequence of [3, Theorem 1].

COROLLARY. *Let E be a Fréchet space, χ be a measure of noncompactness of K, $U \subset K$ be an open neighborhood of* 0 (*in K*) *and* $H: [0, 1] \times \mathrm{cl}_K(U) \to 2^K$ *be u.s.c. Suppose that*

(12) *for* $t \in [0, 1]$ *and* $x \in \mathrm{cl}_K(U)$ *the topological space* $H(t, x)$ *is compact and acyclic* (*i.e.,* $H(t, x)$ *has the same Čech homology with rational coefficients as a one point space*),

(13) $x \notin H(t, x)$ *for* $t \in [0, 1]$ *and* $x \in \partial_K U$,

(14) $X \subset \mathrm{cl}_K(U)$ *and* $\chi(X) \leqslant \chi(H([0, 1] \times X))$ *imply that X is relatively compact and*

(15) $\mu x \notin H(1, x)$ *for* $\mu > 1$ *and* $x \in \partial_K U$.

Then there is $x \in \mathrm{cl}_K(U)$ *such that* $x \in H(0, x)$.

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