PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 72, Number 2, November 1978

POWER BOUNDED STRICTLY CYCLIC OPERATORS

ERIK J. ROSENTHAL

ABSTRACT. We show that a power bounded hereditarily strictly cyclic operator on Hilbert space is similar to a contraction. We also show that certain "almost unitary" operators are not strictly cyclic.

Recall that the operator T is *power bounded* if there exists a positive real number M such that $\|T^n\| \leqslant M$ for $n = 1, 2, 3, \ldots$. Sz.-Nagy **[16]** proved that every compact power bounded operator is similar to a contraction and asked whether the hypothesis of compactness can be removed. Foguel **[3]** (see also Halmos **[5]**) showed that it cannot be removed–there is a power bounded operator that is not similar to a contraction. (It is still not known if every polynomially bounded operator is similar to a contraction. See Ghatage **[4]** for some related results.)

Given an operator T, let $\mathcal{A}(T)$ denote the uniformly closed algebra generated by T and the identity. Lambert **[11]** defined T to be *strictly cyclic* if there exists a vector x_0 such that $\{Ax_0\colon A \in \mathcal{A}(T)\}$ is the entire space. T is *hereditarily strictly cyclic* if its restriction to every invariant subspace is strictly cyclic. There are a number of known results about strictly cyclic operators–see **[6]**, **[7]**, **[8]**, **[10]**, **[11]**, **[12]**, **[13]**, **[14]**.

In this paper we prove that a power bounded hereditarily strictly cyclic operator is similar to a contraction. We require three lemmas.

LEMMA 1. *If T is a strictly cyclic operator, and if $\mathfrak{M}$ is an invariant subspace of T, then the compression of T to $\mathfrak{M}^\perp$ is strictly cyclic.*

PROOF. Let P be the projection onto $\mathfrak{M}^\perp$, and let e be a strictly cyclic vector for T; i.e.,

$$\{Ae\colon A \in \mathcal{A}(T)\} = \mathcal{H}.$$

Let $f = Pe$ and $m = e - f$. Then it suffices to show that

$$\{PAf\colon A \in \mathcal{A}(T)\} = \mathfrak{M}^\perp.$$

Note that $PAm = 0$ for every A in $\mathcal{A}(T)$. Now, given $x \in \mathfrak{M}^\perp$, choose A in $\mathcal{A}(T)$ such that $x = Ae$. Then

$$PAf = PA(f + m) = PAe = Px = x,$$

and the proof is complete.

LEMMA 2. *If T is hereditarily strictly cyclic and power bounded, and if*

Received by the editors August 23, 1977.
AMS (MOS) subject classifications (1970). Primary 47B99.

$\sigma(T) = \{\lambda_0\}$, *where* $|\lambda_0| = 1$, *then T acts on a one-dimensional space.*

PROOF. By replacing T by $\frac{1}{\lambda_0}T$, we may assume that $\lambda_0 = 1$. Lambert **[11]** proved that T strictly cyclic implies that every point in the spectrum of T is compression spectrum. Thus, 1 is an eigenvalue for T^*. So choose a unit vector e such that $T^*e = e$, and let

$$T = \begin{pmatrix} 1 & 0 \\ A & B \end{pmatrix}$$

be the decomposition of T with respect to the decomposition of $\mathcal{H} = \bigvee\{e\} \oplus \{e\}^\perp$. We claim that $\{e\}^\perp = \{0\}$. If not, since T is hereditarily strictly cyclic and $B = T|\{e\}^\perp$, B is strictly cyclic. Also, $\sigma(B) = \{1\}$, so there is a unit vector $f \perp e$ with $B^*f = f$. Then decompose T with respect to the decomposition $\mathcal{H} = \bigvee\{e, f\} \oplus \{e, f\}^\perp$:

$$T = \begin{pmatrix} \begin{matrix} 1 & 0 \\ \lambda & 1 \end{matrix} & 0 \\ C & D \end{pmatrix}$$

Since T is power bounded, $\left(\begin{smallmatrix} 1 & 0 \\ \lambda & 1 \end{smallmatrix}\right)$ is power bounded. Since $\left(\begin{smallmatrix} 1 & 0 \\ \lambda & 1 \end{smallmatrix}\right)^n = \left(\begin{smallmatrix} 1 & 0 \\ n\lambda & 1 \end{smallmatrix}\right)$, λ must be 0. Since $\{e, f\}^\perp \in \operatorname{Lat} T$, $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ is strictly cyclic by Lemma 1. But the identity is strictly cyclic only on a one-dimensional space, so $\{e\}^\perp$ must be $\{0\}$.

We should perhaps note that the above lemma implies the following well known fact.

THEOREM 1 (SZ.-NAGY). *A power bounded operator acting on a finite dimensional space is similar to a contraction.*

PROOF. By the Jordan canonical form theorem, it suffices to prove the theorem for a single Jordan block, so consider an operator

$$T = \begin{pmatrix} \gamma & 1 & & & \bigcirc \\ & \gamma & 1 & & \\ & & \ddots & \ddots & \\ & \bigcirc & & \ddots & 1 \\ & & & & \gamma \end{pmatrix}$$

Since T is power bounded and $\|T^n\| \geqslant |\gamma|^n$, $|\gamma| \leqslant 1$. If $|\gamma| = 1$, then Lemma 2 implies that T acts on a one-dimensional space, and we are done. If $|\gamma| < 1$, then T is similar to

$$S = \begin{pmatrix} \gamma & (1 - |\gamma|) & & & \bigcirc \\ & \gamma & (1 - |\gamma|) & & \\ & & \ddots & \ddots & \\ & \bigcirc & & \ddots & (1 - |\gamma|) \\ & & & & \gamma \end{pmatrix}$$

(by the Jordan canonical form theorem). Then $\|S\| \leqslant \|\gamma I\| + \|(1 - |\gamma|)S_0\|$ where S_0 is a shift all of whose weights are 1. Thus $\|S\| \leqslant |\gamma| + 1 - |\gamma| = 1$.

The next lemma is a slight variant of Theorem 4 of Barnes **[1]** (which is the case where $\|T\| = 1$).

LEMMA 3. *If T is strictly cyclic and power bounded, and if $\lambda \in \sigma(T)$, $|\lambda| = 1$, then λ is an isolated point of $\sigma(T)$.*

PROOF. Suppose that $\|T^n\| \leqslant M$ for all n, and that λ is as stated. By replacing T by $\frac{1}{\lambda}T$, we may assume that $\lambda = 1$. Let $S = \frac{1}{2}(T + 1)$. Then for each n,

$$\|S^n\| = \frac{1}{2^n}\left\|(T + 1)^n\right\| \leqslant \frac{1}{2^n} \cdot M(1 + 1)^n = M.$$

Since $\{S^n\}$ is bounded, some subsequence $\{S^{n_i}\}$ converges weakly to some operator R.

The algebra $\mathcal{A}(T)$ is maximal abelian by a result of Lambert **[10]**. The spectrum of each element of $\mathcal{A}(T)$ is the same as its spectrum relative to the Banach algebra $\mathcal{A}(T)$ (cf. **[15**, p. 4]). Thus, if $\mathfrak{M}$ is the set of nonzero complex homomorphisms of $\mathcal{A}(T)$, then

$$\sigma(T - R) = \{\phi(T - R) \colon \phi \in \mathfrak{M}\}.$$

Now, if $\phi(T) \neq 1$, $|\phi(T) + 1| < 2$ since $|\phi(T)| \leqslant 1$. Hence $|\phi(S)| < 1$, and $\phi(S^n) \to 0$. Lambert **[11**, Theorem 1.7] proved that every linear functional on $\mathcal{A}(T)$ is weakly continuous. So $\phi(T) \neq 1$ implies $\phi(R) = 0$, and $\phi(T) = 1$ implies $\phi(R) = 1$ for every $\phi \in \mathfrak{M}$. Hence

$$\sigma(T - R) = \{\phi(T) - \phi(R) \colon \phi \in \mathfrak{M}\} = \big(\{\phi(T) \colon \phi \in \mathfrak{M}\} \setminus \{1\}\big) \cup \{0\}.$$

Since $\sigma(T - R)$ must be closed, 1 must be an isolated point of $\sigma(T)$; (if $\{\lambda_n\} \to 1$ and $\{\lambda_n\} \subset \sigma(T) \setminus \{1\}$, then $\{\lambda_n\} \subset \sigma(T - R)$, and since $1 \notin \sigma(T - R)$, $\sigma(T - R)$ would not be closed).

THEOREM 2. *A power bounded hereditarily strictly cyclic operator is similar to a contraction.*

PROOF. Let T be power bounded and hereditarily strictly cyclic. The spectral radius of T is at most 1; by the preceding lemma, $\sigma(T) \cap \{z \colon |z| = 1\}$ is some finite set $\{\lambda_1, \ldots, \lambda_n\}$. (If $\sigma(T) \cap \{z \colon |z| = 1\} = \varnothing$, the first part of the proof is not required.) For each j let

$$P_j = \frac{1}{2\pi i} \int_{\Gamma_j} (z - T)^{-1}\, dz$$

where Γ_j is a circle about λ_j such that $\Gamma_j \cap \sigma(T) = \varnothing$ and the interior of Γ_j meets $\sigma(T)$ only in $\{\lambda_j\}$. Then each P_j is a Riesz projection of T (cf. **[15**, Chapter 2]). If $\mathfrak{M}_j = \mathfrak{R}(P_j)$, then the well-known theorem of Riesz (cf. **[15**, p. 31]) implies that $\mathfrak{M}_j$ is invariant under T, and that $\sigma(T|\mathfrak{M}_j) = \{\lambda_j\}$ for each j. Clearly $T|\mathfrak{M}_j$ is also power bounded, so $\mathfrak{M}_j$ is one-dimensional by Lemma

2. Now if $\mathfrak{M} = \bigvee_{j=1}^{n} \mathfrak{M}_j$, then $\mathfrak{M}$ is a finite-dimensional invariant subspace of T, so $T|\mathfrak{M}$ is similar to a contraction by Theorem 1. Also, $\mathfrak{M}$ has an invariant complement $\mathfrak{N}$, ($\mathfrak{N}$ is the null space of $P_1 + P_2 + \ldots + P_n$). Evidently $\sigma(T|\mathfrak{N}) \subset \{z: |z| < 1\}$. Thus, by Rota's theorem (cf. [**15**, Theorem 3.28]) $T|\mathfrak{N}$ is similar to a part of the backwards shift of multiplicity the dimension of $\mathfrak{N}$. Obviously every such part is a contraction.

Now, $\mathfrak{H} = \mathfrak{M} \oplus \mathfrak{N}$, and $T|\mathfrak{M}$ and $T|\mathfrak{N}$ are each similar to a contraction. Since T is similar to $(T|\mathfrak{M}) \oplus (T|\mathfrak{N})$, it follows that T is similar to a contraction.

The next corollary is immediate. It contains the fact that power bounded operators on finite-dimensional spaces are similar to contractions by the Jordan canonical form theorem.

COROLLARY. *A power bounded operator which is the (not necessarily orthogonal) direct sum of a finite number of hereditarily strictly cyclic operators is similar to a contraction.*

Barnes [**1**, Theorem 8] proves that no hyponormal operator is strictly cyclic on an infinite-dimensional space. In particular, there are no strictly cyclic unitary operators on infinite-dimensional spaces. The next theorem is a mild generalization of this fact about unitary operators.

THEOREM 3. *If T is strictly cyclic, power bounded, and invertible, and if there is a $k \geqslant 1$ such that*

$$\lim_{n\to\infty} \frac{1}{n^k} \|T^{-n}\| = 0,$$

then T acts on a finite-dimensional space.

PROOF. Since T is power bounded, $\sigma(T) \subset \{z: |z| \leqslant 1\}$. Choose $M > 0$ such that $\|T^{-n}\| \leqslant M \cdot n^k$. Then the spectral radius of T^{-1} is at most $\lim_{k\to\infty} M^{1/n}(n^k)^{1/n} = 1$, so $\sigma(T^{-1}) \subset \{z: |z| \leqslant 1\}$. Since $\sigma(T^{-1}) = \{1/z: z \in \sigma(T)\}$, it follows that $\sigma(T) \subset \{z: |z| = 1\}$. Now Lemma 3 implies that $\sigma(T) = \{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ for complex numbers λ_i of modulus 1. As in the proof of Theorem 2, let $\mathfrak{M}_i$ be the Riesz subspace such that $\sigma(T|\mathfrak{M}_i) = \{\lambda_i\}$. Since each $\mathfrak{M}_i$ has an invariant complement, $\mathfrak{M}_i$ is invariant under T^{-1} for each i. Thus,

$$\lim_{n\to\infty} \frac{1}{n^k} \|(T|\mathfrak{M}_i)^n\| = \lim_{n\to\infty} \frac{1}{n^k} \|(T|\mathfrak{M}_i)^{-n}\| = 0.$$

These growth conditions together with the fact that $\sigma(T|\mathfrak{M}_i)$ is the singleton $\{\lambda_i\}$ imply that $(T|\mathfrak{M}_i - \lambda_i)^k = 0$ by a theorem of Hille (see [**9**. p. 128]). This implies T is an algebraic operator, so each of its cyclic invariant subspaces is finite-dimensional. Since T is cyclic, it acts on a finite-dimensional space.

REFERENCES

1. B. Barnes, *Operators with a strictly cyclic vector*, Proc. Amer. Math. Soc. **41** (1973), 480–486.

2. M. Embry, *Maximal invariant subspaces of strictly cyclic operator algebras*, Pacific J. Math. **49** (1973), 45–50.

3. S. R. Foguel, *A counterexample to a problem of Sz.-Nagy*, Proc. Amer. Math. Soc. **15** (1964), 788–790.

4. P. Ghatage, *Generalized algebraic operators*, Proc. Amer. Math. Soc. **52** (1975), 232–236.

5. R. P. Halmos, *On Foguel's answer to Sz.-Nagy's question*, Proc. Amer. Math. Soc. **15** (1964), 791–793.

6. J. H. Hedlund, *Strongly strictly cyclic weighted shifts*, Proc. Amer. Math. Soc. **57** (1976), 119–121.

7. D. A. Herrero, *Transitive operator algebras containing a subalgebra of finite strict multiplicity*, Rev. Un. Mat. Argentina **26** (1972), 77–83.

8. ______, *Operator algebras of finite strict multiplicity*, Indiana Math. J. **22** (1972), 13–24.

9. E. Hille and R. S. Phillips, *Functional analysis and semi-groups*, Amer. Math. Soc. Colloq. Publ., vol. 31, Amer. Math. Soc., Providence, R. I., 1957.

10. E. Kerlin and A. L. Lambert, *Strictly cyclic shifts on lp*, Acta Sci. Math. (Szeged) **35** (1973), 87–94.

11. A. L. Lambert, *Strictly cyclic operator algebras*, Dissertation, Univ. of Michigan, 1970.

12. ______, *Strictly cyclic weighted shifts*, Proc. Amer. Math. Soc. **29** (1971), 331–336.

13. ______, *Strictly cyclic operator algebras*, Pacific J. Math. **39** (1971), 717–726.

14. A. L. Lambert and T. R. Turner, *The double commutant of invertibility weighted shifts*, Duke Math. J. **39** (1972), 385–389.

15. H. Radjavi and P. M. Rosenthal, *Invariant subspaces*, Springer-Verlag, Berlin and New York, 1973.

16. B. Sz.-Nagy, *Completely continuous operators with uniformly bounded iterates*, Publ. Math. Debrecen **4** (1959), 82–92.

Department of Mathematics, State University of New York, College at New Paltz, New Paltz, New York 12562