PERIODIC SOLUTIONS OF
PERTURBED CONSERVATIVE SYSTEMS

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Abstract. The existence of $2\pi$-periodic solutions to the system $x'' + \nabla G(x) = p(t, x)$, $p$ being $2\pi$-periodic in $t$, is established under conditions at infinity on the Hessian matrix of $G$. The condition used is weaker than earlier known conditions of a similar nature.

Introduction and preliminaries. This note concerns the differential equation

$$x'' + \nabla G(x) = p(t, x), \tag{I}$$

where $G \in C^2(R^n, R)$ and $p \in C(R \times R^n, R^n)$ is $2\pi$-periodic in $t$ for each fixed $x \in R^n$. We give conditions “at infinity” on the Hessian matrix of $G(x)$ which imply the existence of at least one $2\pi$-periodic solution of (I) when $p(t, x)$ is uniformly bounded.

The existence of $2\pi$-periodic solutions to the equation

$$x'' + \nabla G(x) = f(t) \tag{II}$$

under condition (1.1) below has been the object of several interesting papers. Loud [8] initiated these studies with an investigation of a scalar version of (II). Leach in [7] extended these results. In [6] Lazer and Sánchez, using the Cesari alternative method, proved the following theorem.

**Theorem A [Lazer-Sánchez].** Let $f \in C(R, R^n)$ be $2\pi$-periodic. If $G \in C^2(R^n, R)$ and there exist an integer $\bar{n}$ and numbers $p$ and $q$ such that

$$\bar{n}^2 < p < q < (\bar{n} + 1)^2$$

and if

$$pl \leq \left( \frac{\partial^2 G(a)}{\partial x_i \partial x_j} \right) qI \tag{1.1}$$

for all $a \in R^n$, then there exists a $2\pi$-periodic solution of (II).

In other papers Lazer [5], Ahmad [1], and Kannan [3], and others, have extended these studies to show both existence and uniqueness for wider classes of systems. Recently Mawhin [9] has given a proof of Theorem A (also showing uniqueness) based on an abstract result of his which is in turn based upon a simple and elegant application of the contraction mapping principle. All of these papers have used condition (1.1) or a more general version of (1.1). Here we weaken (1.1) to hold “at infinity” and at the same time allow

Presented to the Society, January 4, 1978; received by the editors November 7, 1977.

AMS (MOS) subject classifications (1970). Primary 34C25, 47H15; Secondary 34B15.

Key words and phrases. Periodic solutions, boundary value problems, nonlinear system of ordinary differential equations, Hilbert space methods.

The author received support from a Pan American University Faculty Research Grant while writing this paper.

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our perturbation to depend upon \( x \). Our methods are most closely related to those of Mawhin.

In this note \( R \) denotes the real numbers, \(|x|\) denotes the Euclidean norm of \( x \in \mathbb{R}^n \), \( n \geq 1 \) and \(|A|\) will be used for the norm of a matrix \( A \). Also \( I \) denotes either the \( n \times n \) identity matrix or the identity map on a Hilbert space. If \( A \) and \( B \) are two real \( n \times n \) matrices by \( A \prec B \) we mean \( B - A \) is nonnegative definite.

2. Statement and proof of the theorem.

\textbf{Theorem 1.} Let \( G \in C^2(\mathbb{R}^n, \mathbb{R}) \) and \( p \in C(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}^n) \) with \( p(t + 2\pi, x) = p(t, x) \) for all \((t, x)\) and \( p(t, x) \) uniformly bounded. Suppose there exist an integer \( \tilde{n} \) and positive numbers \( p, q, r \) with \( n^2 < p < q < (\tilde{n} + 1)^2 \) such that whenever \( a \in \mathbb{R}^n \) and \(|a| > r\)

\[
pi < \left( \frac{\partial^2 G(a)}{\partial x_i \partial x_j} \right) < q1. \tag{III}
\]

Then there is at least one \( 2\pi \)-periodic solution of (I).

\textbf{Proof.} Without loss of generality we may assume grad \( G(0) = 0 \) since otherwise we could subtract grad \( G(0) \) from each side of the equation (I).

Let \( S = L^2((0, 2\pi), \mathbb{R}^n) \) be the Hilbert space of square integrable \( \mathbb{R}^n \) valued functions with the usual inner product

\[
(u, v) = \int_0^{2\pi} \langle u(t), v(t) \rangle \, dt
\]

and norm denoted by \(|| \cdot |||\). Here \( \langle \cdot, \cdot \rangle \) denotes the usual inner product on \( \mathbb{R}^n \).

Let

\[
D = \{ u \in S: u, u' \in AC, u'' \in L^2, u(0) = u(2\pi) \text{ and } u'(0) = u'(2\pi) \}
\]

and define \( L: D \to S \) by \( Lu = u'' \). Here AC means absolutely continuous.

It follows from (III), the fact that the Hessian \( H(x) \) is symmetric, and the continuity of the Hessian \( H(x) = (\partial^2 G(x)/\partial x_i \partial x_j) \) on \(|x| < r\) that grad \( G \) is globally Lipschitzian and there is a number \( a > 0 \) such that \(|\text{grad } G(x)| < a|x| \) for all \( x \in \mathbb{R}^n \). Therefore, if we define an operator \( N \) on \( S \) by \( N(u)(t) = \text{grad } G(u(t)) \) for \( u \in S \) and \( t \in [0, 2\pi] \) the operator \( N \) will map \( S \) continuously into itself. We define \( M: S \to S \) by \( Mu(t) = p(t, u(t)) \) for \( u \in S \) and \( 0 < t < 2\pi \). It is shown in [4, p. 22] that the continuity and boundedness of \( P \) implies that \( M \) is continuous and maps \( S \) into itself. Any \( 2\pi \)-periodic solution of (I) is then a solution of the equation

\[
Lx + Nx = Mx \tag{2.1}
\]

in \( S \), and conversely.

Let \( c = (p + q)/2 \); then \(-c \in \rho(L)\) (the spectrum of \( L \) is \( \{ -n^2: n \in \mathbb{Z} \} \)) and \((L + ci)^{-1}\) exists as a bounded linear operator on \( S \) and it follows from known results concerning Green’s functions (generalized to the case of
uncoupled equations in $R^n$) that $(L + cI)^{-1}$ is compact [2, p. 192]. Equation (2.1) is equivalent to

$$x = (L + cI)^{-1}[M + cI - N]x. \tag{2.2}$$

We will use the Schauder fixed point theorem [11, p. 25] to show that (2.2) has a solution. The operator $M + cI - N$ is continuous and maps bounded sets in $S$ into bounded sets. The compactness of $(L + cI)^{-1}$ now implies that the operator $T$ defined by $T = (L + cI)^{-1}(M + cI - N)$ is completely continuous. If we can show that $T$ maps some closed ball in $S$ into itself we are done.

We first compute $||(L + cI)^{-1}||$. Because $L$ is selfadjoint and the interval $(-(\bar{n} + 1)^2, -\bar{n}^2)$ contains no numbers in the spectrum of $L$ we have as in [9]:

$$||(L + cI)^{-1}|| = \max \left\{ \frac{1}{(c - \bar{n}^2)}, \frac{1}{((\bar{n} + 1)^2 - c)} \right\} = \left[ \min \{ \ldots \} \right]^{-1}.$$

Now let $u \in S$. We estimate $||(M + cI - N)u||$. Let $H(a) = (\partial^2G(a)/\partial x_i \partial x_j)$ and $m$ a number with $m > |p(t, x)|$ for all $t, x$. We have

$$||(M + cI - N)u|| < ||Mu|| + ||(N - cI)u|| < \sqrt{2\pi} m + ||(N - cI)u||. \tag{2.4}$$

Further:

$$||Nu - cu||^2 = \int_0^{2\pi} |\text{grad } G(u(t)) - cu(t)|^2 dt$$

$$= \int_0^{2\pi} \left[ \int_0^1 (H(\lambda u(t)) - cI) d\lambda \right] |u(t)|^2 dt$$

$$< \int_0^{2\pi} \left[ \int_0^1 |H(\lambda u(t)) - cI| d\lambda \right]^2 |u(t)|^2 dt$$

by Taylor's theorem and since grad $G(0) = 0$.

We make the observation that whenever $|\lambda u(t)| > r$ we have as in [9] by the symmetry of the matrix $(\partial^2G(\lambda u(t))/\partial x_i \partial x_j) = H(\lambda u(t))$:

$$|H(\lambda u(t)) - cI| = \sup_{|y| = 1} \langle H(\lambda u(t))y - cy, y \rangle$$

$$< \max\{q - c, c - p\} = \beta. \tag{2.5}$$

Let $z(\lambda u(t)) = |H(\lambda u(t)) - cI|$ and choose $\varepsilon > 0$. Let $E_1 = \{\lambda, t): |\lambda u(t)| < r \text{ and } 0 < \lambda < \varepsilon\}$, $E_2 = \{\lambda, t): |\lambda u(t)| < r \text{ and } \varepsilon < \lambda < 1\}$, and $E_3 = \{\lambda, t): |\lambda u(t)| > r\}$. We have:
\[
\|Nu - cu\|^2 < \int_0^{2\pi} \left[ \int_0^1 z(\lambda u(t)) d\lambda \right]^2 |u(t)|^2 dt \\
= \int \left[ \int_{E_1} z(\lambda u(t)) d\lambda \right]^2 |u(t)|^2 dt \\
+ \int \left[ \int_{E_2} z(\lambda u(t)) d\lambda \right]^2 |u(t)|^2 dt \\
+ \int \left[ \int_{E_3} z(\lambda u(t)) d\lambda \right]^2 |u(t)|^2 dt \\
< \varepsilon^2 k^2 \|u\|^2 + k^2 \int_{\{t: |u(t)| < r/\varepsilon\}} |u(t)|^2 dt + \beta^2 \|u\|^2 \\
< \varepsilon^2 k^2 \|u\|^2 + k^2 r^2/\varepsilon^2 + \beta^2 \|u\|^2
\]

(2.6)

where \( k = \sup\{|H(x) - cl|: |x| < r\} \). If \( \|u\| > kr/\varepsilon^2 \) we have from (2.6):

\[
\|Nu - cu\|^2 < \varepsilon^2 (k^2 + 1) \|u\|^2 + \beta^2 \|u\|^2,
\]

\[
\|Nu - cu\| < \left[ \varepsilon^2 (k^2 + 1) + \beta^2 \right]^{1/2} \|u\|. \quad (2.7)
\]

Since \( \beta = \max\{(q - c), (c - p)\} \) and

\[
\|(L + cl)^{-1}\| = \left[ \min\{\ldots\} \right]^{-1}
\]

and \( c = (p + q)/2 \) with \( \tilde{n}^2 < p < q < (\tilde{n} + 1)^2 \) it is clear that

\[
\beta \|(L + cl)^{-1}\| < 1. \quad (2.8)
\]

Thus we may choose \( \varepsilon = \varepsilon_1 \) so small that whenever \( \|u\| > kr/\varepsilon^2 \) we have by (2.7) and (2.8) that there is a number \( d, 0 < d < 1 \), such that

\[
\|(L + cl)^{-1}\| \|Nu - cu\| < d \|u\|. \quad (2.9)
\]

We can now show that the operator \( T = (L + cl)^{-1}(M + cl - N) \) maps a closed ball of the form \( B_n = \{u \in S: \|u\| < n\} \), \( n \) a positive integer, into itself. If not, then we can find a sequence \( \{x_n\} \) in \( S \) with \( x_n \in B_n \) and \( \|Tx_n\| > n \). The sequence \( \{x_n\} \) must tend to infinity in norm, since otherwise \( T \) would be mapping a bounded set onto an unbounded one. Hence \( \|x_n\| \to \infty \) and \( \|x_n\| > rk/\varepsilon_1 \) for all but finitely many \( n \). By (2.4) and (2.9) we have for large \( n \):

\[
\|x_n\| < n < \|Tx_n\| < d \|x_n\| + \sqrt{2\pi} \ m \|(L + cl)^{-1}\|
\]

and

\[
\|x_n\| < (1 - d)^{-1} \sqrt{2\pi} \ m \|(L + cl)^{-1}\|.
\]

This contradicts the unboundedness of the \( \{x_n\} \). Hence \( T \) maps some ball \( B_k \) into itself. By the Schauder theorem there exists \( x_0 \in B_k \) with \( x_0 = Tx_0 \). Hence \( x_0 \in D(L) \) and
\[ Lx_0 = -Nx_0 + Mx_0. \]

Since \( x_0(t) \) is continuous on \([0, 2\pi]\) and

\[ x_0''(t) = -\nabla G(x_0(t)) + p(t, x_0(t)) \]

it follows that \( x_0''(t) \) is continuous on \([0, 2\pi]\). The function \( x_0(t) \) may now be extended periodically to all of \( R \). This extension is clearly a periodic solution of (I) on all of \( R \). This completes the proof of the theorem.

**Remark 1.** The uniform boundedness of \( p(t, x) \) was not essential, and it is clear that the theorem remains true if \( p(t, x) \) is sublinear, i.e., if

\[ \lim_{|x| \to \infty} \frac{|p(t, x)|}{|x|} = 0, \]

the convergence being uniform in \( t \).

**Remark 2.** This method is not restricted to the periodic problem, but could also be used to handle other problems.

**Remark 3.** Reissig [10] has extended Mawhin’s approach to the equation

\[ x'' + Cx' + \nabla G(x) = e(t) \]

with \( C \) symmetric. His results can be combined with the methods of this paper to give similar results to our Theorem 1 for the equation

\[ x'' + Cx' + \nabla G(x) = p(t, x). \]

The author is indebted to the referee for this observation.

**References**


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