A THEOREM OF BEURLING AND TSUJI IS BEST POSSIBLE

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Abstract. We shall show that Beurling-Tsuji's theorem (see Theorem A) is, in a sense, best possible. For each pair $a, b \in (0, + \infty)$ there exists a function $f$ holomorphic in $\{|z| < 1\}$ such that the Euclidean area of the Riemannian image of each non-Euclidean disk of non-Euclidean radius $a$ is bounded by $b$, and such that $f$ has finite angular limit nowhere on the unit circle.

1. Introduction. Let $D = \{|z| < 1\}$, and let $\Gamma = \{|z| = 1\}$. For a function $f$ holomorphic in $D$, and for a subset $E$ of $D$ we use the notation

$$A(E, f) = \iint_{E} |f'(z)|^2 \, dx \, dy, \quad z = x + iy.$$ 

The following theorem is due to A. Beurling and M. Tsuji.

**Theorem A** ([1], [4], see [5, Theorem VIII.49, p. 344]). Let $f$ be a function holomorphic in $D$ with $A(D, f) < + \infty$. Then $f$ has a finite angular limit at each point of $\Gamma$ except for a set of zero logarithmic capacity.

We shall show that extensions of Theorem A are, in a sense, impossible. Consider the non-Euclidean hyperbolic metric

$$\sigma(w, z) = \frac{1}{2} \log \frac{|1 - \overline{z}w| + |z - w|}{|1 - \overline{z}w| - |z - w|}, \quad z, w \in D,$$

to define

$$H(z, a) = \{w \in D; \sigma(w, z) < a\}, \quad z \in D, a \in (0, + \infty].$$

We let $F(a, b)$ be the family of all holomorphic functions $f$ in $D$ such that, for each $z \in D$, $A(H(z, a), f) < b$, where $a \in (0, + \infty]$ and $b \in (0, + \infty)$. Then, $f$ of Theorem A belongs to $F(+ \infty, b)$ with $b = A(D, f)$.

**Theorem 1.** Given $a \in (0, + \infty)$ and $b \in (0, \infty)$, we may find $f \in F(a, b)$ such that neither $\Re f$ nor $\Im f$ has a finite angular limit at any point of $\Gamma$.

Thus, $f$ has not a finite angular limit at any point of $\Gamma$.

2. Bloch function. A function $f$ in $D$ is called Bloch [3] if $f$ is holomorphic in $D$ and if

$$\beta(f) = \sup_{z \in D} (1 - |z|^2)|f'(z)| < + \infty.$$ 

Let $B(c)$ be the family of all Bloch functions $f$ with $\beta(f) < c$, $c \in (0, + \infty)$.

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A THEOREM OF BEURLING AND TSUJI IS BEST POSSIBLE 287

Theorem 2. (2.1) If \(a \in (0, + \infty)\), then each \(f \in B(c)\) is a member of \(F(a, b)\) with
\[ b = \pi c^2 p(a)^2 / \left(1 - p(a)^2\right), \]
where
\[ p(a) = (e^{2a} - 1) / (e^{2a} + 1). \]

(2.2) If \(a \in (0, + \infty)\), \(b \in (0, + \infty)\), and if \(f \in F(a, b)\), then \(f \in B(c)\) with
\[ c^2 = b / \left[\pi p(a)^2\right] \quad (p (+ \infty) = 1). \]

Proof. (2.1) Since
\[ |f'(w)| < c(1 - |w|^2)^{-1}, \quad w \in D, \]
it follows that, for each \(z \in D\),
\[ A(H(z, a), f) < c^2 \int_{H(z, a)} (1 - |w|^2)^{-2} dx \, dy \]
\[ = c^2 \int_{|w| < p(a)} (1 - |w|^2)^{-2} dx \, dy = b \quad (w = x + iy), \]
because \((1 - |w|^2)^{-2} dx \, dy\) is invariant under non-Euclidean transformations.

(2.2) Set \(p = p(a)\), and for each fixed \(z \in D\), set
\[ g(w) = f((pw + z) / (1 + pw)), \quad w \in D. \]
Then
\[ \pi p^2 (1 - |z|^2)^2 |f'(z)|^2 = \pi |g'(0)|^2 < A(D, g) = A(H(z, a), f) < b. \]
Therefore, \(\beta(f) < c\), whence \(f \in B(c)\).

3. Proof of Theorem 1. We shall make use of the two lemmata due to P. A. Lappan:

Lemma 1 [2, p. 113]. There exists a holomorphic function \(g\) in \(D\), satisfying
\[ \sup_{z \in D} (1 - |z|) |g(z)| < 2, \quad (3.1) \]
and
\[ \limsup_{0 < r \to 1^-} (1 - r) |g(r \xi)| > 0 \quad (3.2) \]
for each \(\xi \in \Gamma\).

Lemma 2 [2, Lemma 3]. Let \(f\) be a holomorphic function in \(D\) such that
\[ \limsup_{0 < r \to 1^-} (1 - r) |f'(r \xi)| > 0 \quad (3.3) \]
at a point \(\xi \in \Gamma\). Then, neither \(\text{Re} f\) nor \(\text{Im} f\) has a finite angular limit at \(\xi\).

We note that our Lemma 2 is worded differently than Lemma 3 of [2], but the content is equivalent.

For the proof of Theorem 1 we choose \(k > 0\) such that
Let $f$ be a function holomorphic in $D$ such that $f' = k^{-1}g$, where $g$ is the function in Lemma 1. Then it follows from (3.1) that $f \in B(c)$ with $c = 4k^{-1}$. It follows from Theorem 2, (2.1), together with (3.4) that $f \in F(a, b)$. It further follows from (3.2) that (3.3) is true at every point $\xi$ of $\Gamma$. Thus, our assertion on the angular limits of $\text{Re}f$ and $\text{Im}f$ follows from Lemma 2.

**REFERENCES**


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