A THEOREM OF BEURLING AND TSUJI IS BEST POSSIBLE

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Abstract. We shall show that Beurling-Tsuji’s theorem (see Theorem A) is, in a sense, best possible. For each pair a, b ∈ (0, + ∞) there exists a function f holomorphic in \(|z| < 1\) such that the Euclidean area of the Riemannian image of each non-Euclidean disk of non-Euclidean radius a, is bounded by b, and such that f has finite angular limit nowhere on the unit circle.

1. Introduction. Let \(D = \{|z| < 1\}\), and let \(\Gamma = \{|z| = 1\}\). For a function f holomorphic in D, and for a subset E of D we use the notation

\[ A(E, f) = \iint_E |f'(z)|^2 \, dx \, dy, \quad z = x + iy. \]

The following theorem is due to A. Beurling and M. Tsuji.

Theorem A ([1], [4], see [5, Theorem VIII.49, p. 344]). Let f be a function holomorphic in D with \(A(D, f) < + \infty\). Then f has a finite angular limit at each point of \(\Gamma\) except for a set of zero logarithmic capacity.

We shall show that extensions of Theorem A are, in a sense, impossible.

Consider the non-Euclidean hyperbolic metric

\[ o(w, z) = -\log \frac{|z - w|}{|1 - zw|}, \quad z, w \in D, \]

to define

\[ H(z, a) = \{w \in D; o(w, z) < a\}, \quad z \in D, a \in (0, + \infty]. \]

We let \(F(a, b)\) be the family of all holomorphic functions f in D such that, for each \(z \in D\), \(A(H(z, a), f) < b\), where \(a \in (0, + \infty]\) and \(b \in (0, + \infty)\). Then, f of Theorem A belongs to \(F(+ \infty, b)\) with \(b = A(D, f)\).

Theorem 1. Given \(a \in (0, + \infty)\) and \(b \in (0, \infty)\), we may find \(f \in F(a, b)\) such that neither Re \(f\) nor Im \(f\) has a finite angular limit at any point of \(\Gamma\).

Thus, f has not a finite angular limit at any point of \(\Gamma\).

2. Bloch function. A function f in D is called Bloch [3] if f is holomorphic in D and if

\[ \beta(f) = \sup_{z \in D} (1 - |z|^2)|f'(z)| < + \infty. \]

Let \(B(c)\) be the family of all Bloch functions f with \(\beta(f) < c\), \(c \in (0, + \infty)\).
Theorem 2. (2.1) If \( a \in (0, + \infty) \), then each \( f \in B(c) \) is a member of \( F(a, b) \) with
\[
b = \pi c^2 p(a)^2 / (1 - p(a)^2),
\]
where
\[
p(a) = (e^{2a} - 1) / (e^{2a} + 1).
\]
(2.2) If \( a \in (0, + \infty] \), \( b \in (0, + \infty) \), and if \( f \in F(a, b) \), then \( f \in B(c) \) with
\[
c^2 = b / \left[ \pi p(a)^2 \right] \quad (p(+\infty) = 1).
\]

Proof. (2.1) Since
\[
|f'(w)| < c(1 - |w|^2)^{-1}, \quad w \in D,
\]
it follows that, for each \( z \in D \),
\[
A(H(z,a),f) < c^2 \int \int_{H(z,a)} (1 - |w|^2)^{-2} \, dx \, dy
\]
\[
= c^2 \int \int_{|w| < p(a)} (1 - |w|^2)^{-2} \, dx \, dy = b \quad (w = x + iy),
\]
because \((1 - |w|^2)^{-2} dx \, dy\) is invariant under non-Euclidean transformations.

(2.2) Set \( p = p(a) \), and for each fixed \( z \in D \), set
\[
g(w) = f((pw + z) / (1 + \bar{p}w)), \quad w \in D.
\]
Then
\[
\pi p^2 (1 - |z|^2)^2 |f'(z)|^2 = \pi |g'(0)|^2 \leq A(D, g) = A(H(z,a),f) \leq b.
\]
Therefore, \( \beta(f) \leq c \), whence \( f \in B(c) \).

3. Proof of Theorem 1. We shall make use of the two lemmata due to P. A. Lappan:

Lemma 1 [2, p. 113]. There exists a holomorphic function \( g \) in \( D \), satisfying
\[
\sup_{z \in D} (1 - |z|) |g(z)| < 2, \quad (3.1)
\]
and
\[
\limsup_{0 < r \to 1^-} (1 - r) |g(r\xi)| > 0 \quad (3.2)
\]
for each \( \xi \in \Gamma \).

Lemma 2 [2, Lemma 3]. Let \( f \) be a holomorphic function in \( D \) such that
\[
\limsup_{0 < r \to 1^-} (1 - r) |f''(r\xi)| > 0 \quad (3.3)
\]
at a point \( \xi \in \Gamma \). Then, neither \( \text{Re } f \) nor \( \text{Im } f \) has a finite angular limit at \( \xi \).

We note that our Lemma 2 is worded differently than Lemma 3 of [2], but the content is equivalent.

For the proof of Theorem 1 we choose \( k > 0 \) such that
\[ k^2 = 16\pi p(a)^2 / \left[ b(1 - p(a)^2) \right]. \] (3.4)

Let \( f \) be a function holomorphic in \( D \) such that \( f' = k^{-1}g \), where \( g \) is the function in Lemma 1. Then it follows from (3.1) that \( f \in B(c) \) with \( c = 4k^{-1} \). It follows from Theorem 2, (2.1), together with (3.4) that \( f \in F(a, b) \). It further follows from (3.2) that (3.3) is true at every point \( \xi \) of \( \Gamma \). Thus, our assertion on the angular limits of \( \text{Re} f \) and \( \text{Im} f \) follows from Lemma 2.

**References**

5. ----, Potential theory in modern function theory, Maruzen, Tokyo, 1959.