MINIMAL $H^2$ INTERPOLATION IN
THE CARATHÉODORY CLASS

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Abstract. For $(c_1, \ldots, c_n)$ in $\mathbb{C}^n$, let $C(c_1, \ldots, c_n)$ denote the class of functions $f(z) = 1 + c_1 z + \cdots + c_n z^n + \sum_{k=n+1}^{\infty} a_k z^k$ which are analytic and satisfy $\text{Re} f(z) > 0$ in the unit disc. The unique function of least $H^2$ norm in $C(c_1, \ldots, c_n)$ is explicitly determined.

In this paper we consider a minimal interpolation problem at the origin, for the class $H^2 \cap C$, where $H^2$ denotes the well-known Hardy space for the unit disc, and $C$ is the Carathéodory class of functions $f(z) = 1 + c_1 z + c_2 z^2 + \cdots$ which are analytic and satisfy $\text{Re} f(z) > 0$ in $|z| < 1$. In particular, given $n$ complex numbers $c_1, \ldots, c_n$, we wish to find the function $f$ in $H^2 \cap C$, of the form

$$f(z) = 1 + c_1 z + \cdots + c_n z^n + \sum_{k=n+1}^{\infty} a_k z^k$$

with least $H^2$ norm.

For each $n$, expansion (1) defines a mapping

$$\nu_n : f \mapsto (c_1, \ldots, c_n)$$

of $C$ onto some compact set $C_n \subset \mathbb{C}^n$, which is called the $n$th coefficient body for $C$. The following result is due to C. Carathéodory and O. Toeplitz (see [2]):

Theorem A. $C_n$ is a convex, compact body in $\mathbb{C}^n$. To each point in the interior of $C_n$ there correspond infinitely many functions in $C$; but each boundary point of $C_n$ corresponds to only one $f$ in $C$. The boundary points correspond to functions of the form

$$f(z) = \sum_{k=1}^{m} \frac{1 + \alpha_k z}{1 - \alpha_k z} \mu_k,$$

where $1 < m < n$; $|\alpha_k| = 1$, $\mu_k > 0$ ($k = 1, \ldots, m$), and $\sum_{k=1}^{m} \mu_k = 1$.

From Theorem A it follows that a function corresponding to a boundary point of $C_n$ has infinite $H^2$ norm. On the other hand, any interior point corresponds to infinitely many functions in $H^2 \cap C$. Indeed, if $(c_1, \ldots, c_n)$ is interior to $C_n$, choose $0 < r < 1$ such that $(b_1, \ldots, b_n) \in C_n$, where $b_k = \ldots$
Let \( g(z) \) be any function in \( C \) which corresponds to \((b_1, \ldots, b_n)\). Then \( f(z) = g(rz) \) is still in \( C \), is now in \( {H^2} \), and corresponds to \((c_1, \ldots, c_n)\).

Thus, our interpolation problem is meaningful precisely for the interior of \( C_n \). If we let \( C(c_1, \ldots, c_n) \) denote the set of all \( f \) in \( C \) of the form (1), we have the following

**Theorem.** For each \((c_1, \ldots, c_n)\) in the interior of \( C_n \), there exists a unique function \( f \) with least \( H^2 \) norm in \( C(c_1, \ldots, c_n) \). This \( f \) is of the form

\[
f(z) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{e^{it} + z}{e^{it} - z} u(t) \, dt,
\]

where \( u(t) = \max(0, P(t)) \), \( P(t) = a_0 + \sum_{k=1}^{n} (a_k \cos kt + b_k \sin kt) \), \((a_k, b_k\) real).

**Proof.** \( C(c_1, \ldots, c_n) \) is a normal and compact family, and \( \|f\|_2 \) is a continuous functional. Thus, we are assured of the existence of an extremal \( f \).

It is classical that functions \( f \) in \( C \) have the Riesz-Herglotz representation

\[
f(z) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{e^{it} + z}{e^{it} - z} d\mu(t),
\]

where \( d\mu(t) \) is a positive measure on \([-\pi, \pi]\) for which \((1/2\pi)\int_{-\pi}^{\pi} d\mu(t) = 1\).

If we write

\[
\hat{\mu}(k) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-ikt} \, d\mu(t) \quad (k = 0, \pm 1, \pm 2, \ldots),
\]

then by comparing Taylor coefficients of both sides of (3), we obtain \( c_k = 2\hat{\mu}(k) \) \((k = 1, 2, \ldots)\). We thus have a one-one correspondence between \( C(c_1, \ldots, c_n) \) and all positive measures \( d\mu(t) \) which satisfy

\[
\hat{\mu}(0) = 1, \quad \hat{\mu}(k) = \frac{1}{2} c_k \quad (k = 1, \ldots, n).
\]

Since

\[
\|f\|_2 = \left( 1 + 2 \sum_{1}^{\infty} |\hat{\mu}(k)|^2 \right)^{1/2},
\]

and the \( L^2 \) norm of \( d\mu(t) \) is \((1 + 2\sum_{1}^{\infty} |\hat{\mu}(k)|^2)^{1/2}\), it follows that \( f \in H^2 \cap C \) corresponds to \( u \in L^2[-\pi, \pi] \), \( u(t) > 0 \), where \( u(t)dt = d\mu(t) \).

Thus, we must find \( u_0 \in L^2[-\pi, \pi] \), \( u_0 > 0 \), which minimizes \( \int_{-\pi}^{\pi} F_j(t, u(t)) \, dt \) under the constraints

\[
\int_{-\pi}^{\pi} F_j(t, u(t)) \, dt = d_j \quad (j = 1, 2, \ldots, 2^{n+1}),
\]
\[ F_0(u, t) = u^2, \]
\[ F_{2k+1}(t, u) = u \cos kt \quad (k = 0, 1, \ldots, n), \]
\[ F_{2k}(t, u) = u \sin kt \quad (k = 1, \ldots, n), \]
\[ d_1 = 2\pi; \quad d_{2k+1} + id_{2k} = \pi c_k \quad (k = 1, \ldots, n). \]

Of course, the existence of an extremal \( f \) implies that a corresponding extremal \( u_0 \) exists. We now need the following

**Lemma.** Suppose \( u_0 \) is the extremal function described above. Then there exist real multipliers \( \lambda_0 = 1, \lambda_1, \ldots, \lambda_{2n+1}, \lambda_{2n+1} \), for which the function

\[ F = \sum_{k=0}^{2n+1} \lambda_k F_k \]

is such that the inequality

\[ F(t, u) > F(t, u_0(t)) \quad (5) \]

holds for all pairs \((t, u) \in E \times [0, \infty), \) where \( E \subseteq [-\pi, \pi] \) contains almost all points of \([-\pi, \pi]\).

Assume for a moment that this Lemma has been established. Then by elementary calculus, the only function \( u_0(t) > 0 \) which minimizes \( F \) in the sense of (5) is

\[ u_0(t) = \max \left( 0, -\frac{1}{2} \lambda_1 - \frac{1}{2} \sum_{k=1}^{n} \lambda_{2k} \sin kt + \lambda_{2k+1} \cos kt \right) \quad \text{a.e.} \quad (6) \]

For convenience, let \( u_0(t) \) denote the continuous representative of the \( L^2 \) equivalence class (6). Then we have the strict inequality \( F(t, u) > F(t, u_0(t)) \) for all \((t, u) \in [-\pi, \pi] \times [0, \infty), u \neq u_0(t). \) If we write

\[ J_k(u) = \int_{-\pi}^{\pi} F_k(t, u(t)) \, dt - d_k \quad (k = 0, 1, \ldots, 2n + 1; d_0 = 0), \]

\[ J = \sum_{k=0}^{2n+1} \lambda_k j_k, \]

then for any \( u(t) > 0 \) satisfying (4), we have \( J(u) = J(u_0). \) Thus, if such a \( u(t) \) differs from \( u_0(t) \) on a set of positive measure, we have \( J_0(u) = J(u) > J(u_0) = J_0(u_0). \) In other words, \( u_0 \) is the unique minimizing function. This in turn implies that the corresponding extremal \( f \), given by (2), is unique. Thus, our Theorem is proven, subject to the Lemma.

**Proof of Lemma.** This Lemma is essentially an \( L^p \) version of Theorem 5.1 in [1, p. 215]. That variational theorem deals with piecewise continuous functions, but in our case, we do not know a priori that the extremal \( u_0 \) is piecewise continuous. In the sequel, we follow the notation of Hestenes [1, Chapters 4, 5].

Let
where \( u_0 \) is a nonnegative \( L^2 \) function which minimizes \( \int_{-\pi}^{\pi} F_0(t, u(t)) \, dt \) under the constraints (4). Let \( E \) be the set of all \( t \in (-\pi, \pi) \) such that \( G_i(t) = F_i(t, u_0(t)) \) \( (i = 0, 1, \ldots, 2n + 1) \). Note that \( [-\pi, \pi] - E \) has measure zero.

Let \( K \) be the class of all vectors \( k = (k^0, k^1, \ldots, k^{2n+1}) \) of the form

\[
k^i = F_i(t, u) - F_i(t, u_0(t)) \quad (i = 0, 1, \ldots, 2n + 1),
\]

where \( t \in E, u > 0 \). For any finite set of vectors \( k_1, \ldots, k_M \) in \( K \), we define \( u(\epsilon) = u(\epsilon, t) \) as in [1, p. 221]. At the point \( \epsilon = 0 \), we have

\[
\frac{\partial J_\epsilon(u(\epsilon))}{\partial \epsilon_j} = k^j_i \quad (i = 0, 1, \ldots, 2n + 1; j = 1, \ldots, M).
\]

Let \( K^* \) denote the cone generated by \( K \), and \( K^- \) the cone generated by the vector \((-1, 0, \ldots, 0)\). \( K^* = K^* - K^- \) denotes the set of all vectors of the form \( k - k^- \), where \( k \in K^* \), \( k^- \in K^- \). Set \( N = 2n + 2 \). Then Lemmas 8.1 and 8.2 follow as in [1, pp. 192–194]. What remains to be proven is that there exists a vector \( k \neq 0 \) which is not in the cone \( K^+ \).

Suppose that \( K^+ \) contained all vectors. Select \( N \) vectors \( k^*_1, \ldots, k^*_N \) in \( K^* \) having the properties described in Lemma 8.2. Since \( K^* \) is generated by \( K \), there exists a finite set of vectors \( k_1, \ldots, k_M \) in \( K \) such that

\[
k^*_j = \sum_{i=1}^{M} k_i a_{ij}, \quad a_{ij} > 0 \quad (j = 1, \ldots, N).
\]

Writing \( \beta = (\beta_1, \ldots, \beta_N) \), \( (\beta_j > 0) \), we define \( \epsilon_i \) by

\[
\epsilon_i = \sum_{j=1}^{N} a_{ij} \beta_j \quad (i = 1, \ldots, M),
\]

or in matrix notation, \( \epsilon = A\beta \).

Writing \( u_A(\beta) = u(A\beta) = u(\epsilon) \), where \( u(\epsilon) \) is as above, we have, at \( \beta = 0, \)

\[
\frac{\partial J_{\epsilon-1}(u_A(\beta))}{\partial \beta_j} = k^*_j \quad (i, j = 1, \ldots, N).
\]

If we now define \( F_i(z, t) \) as in Lemma 8.3 [1, p. 195], (with \( k^*_j \) instead of \( k_j \)), then the \( F_i(z, t) \) will be continuous, and we may proceed as in the proof of Lemma 8.3. Thus, we conclude that there exist multipliers \( \lambda_0 > 0, \lambda_1, \ldots, \lambda_{2n+1} \) such that \( \sum_{0}^{2n+1} \lambda_k k^i > 0 \) for all \( k \in K \). In view of (7), this is exactly inequality (5).

In our specific case, \( \lambda_0 \neq 0 \). Indeed, suppose \( \lambda_0 = 0 \). Then we would have \( F(t, u) = uT(t) \), where \( T \) is a trigonometric polynomial. If \( T(t) \) were negative on some interval, then \( F \) could not have a minimum on that interval. If \( T(t) \) were nonnegative, then the only \( u_0(t) > 0 \) which minimized \( F \) would be \( u_0(t) \equiv 0 \), but this \( u_0 \) does not satisfy the first side condition

\[
\frac{1}{2\pi} \int_{-\pi}^{\pi} u_0(t) \, dt = 1.
\]
Thus, \( \lambda_0 > 0 \), and we may take \( \lambda_0 = 1 \).

**References**


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