PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 72, Number 2, November 1978

MINIMAL H^2 INTERPOLATION IN THE CARATHÉODORY CLASS

E. BELLER AND B. PINCHUK

ABSTRACT. For $(c_1, \ldots, c_n)$ in $\mathbf{C}^n$, let $C(c_1, \ldots, c_n)$ denote the class of functions $f(z) = 1 + c_1 z + \cdots + c_n z^n + \sum_{k=n+1}^{\infty} a_k z^k$ which are analytic and satisfy $\operatorname{Re} f(z) > 0$ in the unit disc. The unique function of least H^2 norm in $C(c_1, \ldots, c_n)$ is explicitly determined.

In this paper we consider a minimal interpolation problem at the origin, for the class $H^2 \cap C$, where H^2 denotes the well-known Hardy space for the unit disc, and C is the Carathéodory class of functions $f(z) = 1 + c_1 z + c_2 z^2 + \cdots$ which are analytic and satisfy $\operatorname{Re} f(z) > 0$ in $|z| < 1$. In particular, given n complex numbers $c_1, \ldots, c_n$, we wish to find the function f in $H^2 \cap C$, of the form

$$f(z) = 1 + c_1 z + \cdots + c_n z^n + \sum_{k=n+1}^{\infty} a_k z^k \tag{1}$$

with least H^2 norm.

For each n, expansion (1) defines a mapping

$$\nu_n : f \to (c_1, \ldots, c_n)$$

of C onto some compact set $C_n \subset \mathbf{C}^n$, which is called the nth coefficient body for C. The following result is due to C. Carathéodory and O. Toeplitz (see [2]):

THEOREM A. *C_n is a convex, compact body in $\mathbf{C}^n$. To each point in the interior of C_n there correspond infinitely many functions in C; but each boundary point of C_n corresponds to only one f in C. The boundary points correspond to functions of the form*

$$f(z) = \sum_{k=1}^{m} \frac{1 + \alpha_k z}{1 - \alpha_k z} \mu_k,$$

where $1 \leqslant m \leqslant n$; $|\alpha_k| = 1$, $\mu_k > 0$ $(k = 1, \ldots, m)$, and $\sum_1^m \mu_k = 1$.

From Theorem A it follows that a function corresponding to a boundary point of C_n has infinite H^2 norm. On the other hand, any interior point corresponds to infinitely many functions in $H^2 \cap C$. Indeed, if $(c_1, \ldots, c_n)$ is interior to C_n, choose $0 < r < 1$ such that $(b_1, \ldots, b_n) \in C_n$, where $b_k =$

Received by the editors November 21, 1977 and, in revised form, January 17, 1978.

AMS (MOS) subject classifications (1970). Primary 30A38, 30A76; Secondary 42A04, 49B35.

Key words and phrases. Positive real part, H^2, minimal interpolation, Riesz-Herglotz representation, coefficient body, multiplier rule.

c_k/r^k $(k = 1, \ldots, n)$. Let $g(z)$ be any function in C which corresponds to $(b_1, \ldots, b_n)$. Then $f(z) \equiv g(rz)$ is still in C, is now in H^2, and corresponds to $(c_1, \ldots, c_n)$.

Thus, our interpolation problem is meaningful precisely for the interior of C_n. If we let $C(c_1, \ldots, c_n)$ denote the set of all f in C of the form (1), we have the following

THEOREM. *For each $(c_1, \ldots, c_n)$ in the interior of C_n, there exists a unique function f with least H^2 norm in $C(c_1, \ldots, c_n)$. This f is of the form*

$$f(z) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{e^{it} + z}{e^{it} - z} u(t)\, dt, \tag{2}$$

where $u(t) = \max(0, P(t))$, $P(t) = a_0 + \sum_{k=1}^{n}(a_k \cos kt + b_k \sin kt)$, $(a_k, b_k$ real).

PROOF. $C(c_1, \ldots, c_n)$ is a normal and compact family, and $\|f\|_2$ is a continuous functional. Thus, we are assured of the existence of an extremal f.

It is classical that functions f in C have the Riesz-Herglotz representation

$$f(z) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{e^{it} + z}{e^{it} - z}\, d\mu(t), \tag{3}$$

where $d\mu(t)$ is a positive measure on $[-\pi, \pi]$ for which $(1/2\pi)\int_{-\pi}^{\pi} d\mu(t) = 1$. If we write

$$\hat{\mu}(k) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-ikt}\, d\mu(t) \qquad (k = 0, \pm 1, \pm 2, \ldots),$$

then by comparing Taylor coefficients of both sides of (3), we obtain $c_k = 2\hat{\mu}(k)$ $(k = 1, 2, \ldots)$. We thus have a one-one correspondence between $C(c_1, \ldots, c_n)$ and all positive measures $d\mu(t)$ which satisfy

$$\hat{\mu}(0) = 1, \quad \hat{\mu}(k) = \tfrac{1}{2} c_k \qquad (k = 1, \ldots, n).$$

Since

$$\|f\|_2 = \left(1 + 4\sum_{1}^{\infty} |\hat{\mu}(k)|^2\right)^{1/2},$$

and the L^2 norm of $d\mu(t)$ is $(1 + 2\sum_1^\infty |\hat{\mu}(k)|^2)^{1/2}$, it follows that $f \in H^2 \cap C$ corresponds to $u \in L^2[-\pi, \pi]$, $u(t) \geqslant 0$, where $u(t)dt = d\mu(t)$.

Thus, we must find $u_0 \in L^2[-\pi, \pi]$, $u_0 \geqslant 0$, which minimizes $\int_{-\pi}^{\pi} F_0(t, u(t))\, dt$ under the constraints

$$\int_{-\pi}^{\pi} F_j(t, u(t))\, dt = d_j \qquad (j = 1, 2, \ldots, 2^{n+1}), \tag{4}$$

where

$$F_0(u, t) = u^2,$$

$$F_{2k+1}(t, u) = u \cos kt \qquad (k = 0, 1, \ldots, n),$$

$$F_{2k}(t, u) = u \sin kt \qquad (k = 1, \ldots, n),$$

$$d_1 = 2\pi; \quad d_{2k+1} + id_{2k} = \pi c_k \qquad (k = 1, \ldots, n).$$

Of course, the existence of an extremal f implies that a corresponding extremal u_0 exists. We now need the following

LEMMA. *Suppose u_0 is the extremal function described above. Then there exist real multipliers $\lambda_0 = 1, \lambda_1, \ldots, \lambda_{2n+1}$, for which the function*

$$F = \sum_{k=0}^{2n+1} \lambda_k F_k$$

is such that the inequality

$$F(t, u) \geqslant F(t, u_0(t)) \tag{5}$$

holds for all pairs $(t, u) \in E \times [0, \infty)$, where $E \subset [-\pi, \pi]$ contains almost all points of $[-\pi, \pi]$.

Assume for a moment that this Lemma has been established. Then by elementary calculus, the only function $u_0(t) \geqslant 0$ which minimizes F in the sense of (5) is

$$u_0(t) = \max\left(0, -\frac{1}{2}\lambda_1 - \frac{1}{2}\sum_{k=1}^{n} \lambda_{2k} \sin kt + \lambda_{2k+1} \cos kt\right) \quad \text{a.e.} \tag{6}$$

For convenience, let $u_0(t)$ denote the continuous representative of the L^2 equivalence class (6). Then we have the strict inequality $F(t, u) > F(t, u_0(t))$ for all $(t, u) \in [-\pi, \pi] \times [0, \infty)$, $u \neq u_0(t)$. If we write

$$J_k(u) = \int_{-\pi}^{\pi} F_k(t, u(t))\, dt - d_k \qquad (k = 0, 1, \ldots, 2n + 1;\ d_0 = 0),$$

$$J = \sum_{k=0}^{2n+1} \lambda_k j_k,$$

then for any $u(t) \geqslant 0$ satisfying (4), we have $J(u) = J(u_0)$. Thus, if such a $u(t)$ differs from $u_0(t)$ on a set of positive measure, we have $J_0(u) = J(u) > J(u_0) = J_0(u_0)$. In other words, u_0 is the *unique* minimizing function. This in turn implies that the corresponding extremal f, given by (2), is unique. Thus, our Theorem is proven, subject to the Lemma.

PROOF OF LEMMA. This Lemma is essentially an L^p version of Theorem 5.1 in [**1**, p. 215]. That variational theorem deals with piecewise continuous functions, but in our case, we do not know *a priori* that the extremal u_0 is piecewise continuous. In the sequel, we follow the notation of Hestenes [**1**, Chapters 4, 5].

Let

$$G_i(t) = \int_{-\pi}^{t} F_i(s, u_0(s))ds \qquad (i = 0, 1, \ldots, 2n + 1),$$

where u_0 is a nonnegative L^2 function which minimizes $\int_{-\pi}^{\pi} F_0(t, u(t))\, dt$ under the constraints (4). Let E be the set of all $t \in (-\pi, \pi)$ such that $G_i'(t) = F_i(t, u_0(t))$ $(i = 0, 1, \ldots, 2n + 1)$. Note that $[-\pi, \pi] - E$ has measure zero.

Let K be the class of all vectors $k = (k^0, k^1, \ldots, k^{2n+1})$ of the form

$$k^i = F_i(t, u) - F_i(t, u_0(t)) \qquad (i = 0, 1, \ldots, 2n + 1), \tag{7}$$

where $t \in E$, $u \geqslant 0$. For any finite set of vectors $k_1, \ldots, k_M$ in K, we define $u(\varepsilon) = u(\varepsilon, t)$ as in [**1**, p. 221]. At the point $\varepsilon = 0$, we have

$$\partial J_i(u(\varepsilon))/\partial \varepsilon_j = k_j^i \qquad (i = 0, 1, \ldots, 2n + 1; j = 1, \ldots, M).$$

Let K^* denote the cone generated by K, and K^- the cone generated by the vector $(-1, 0, \ldots, 0)$. $K^+ = K^* - K^-$ denotes the set of all vectors of the form $k - k^-$, where $k \in K^*$, $k^- \in K^-$. Set $N = 2n + 2$. Then Lemmas 8.1 and 8.2 follow as in [**1**, pp. 192–194]. What remains to be proven is that there exists a vector $k \neq 0$ which is not in the cone K^+.

Suppose that K^+ contained all vectors. Select N vectors $k_1^*, \ldots, k_N^*$ in K^* having the properties described in Lemma 8.2. Since K^* is generated by K, there exists a finite set of vectors $k_1, \ldots, k_M$ in K such that

$$k_j^* = \sum_{i=1}^{M} k_i a_{ij}, \qquad a_{ij} \geqslant 0\ (j = 1, \ldots, N).$$

Writing $\beta = (\beta_1, \ldots, \beta_N)$, $(\beta_j \geqslant 0)$, we define ε_i by

$$\varepsilon_i = \sum_{j=1}^{N} a_{ij}\beta_j \qquad (i = 1, \ldots, M),$$

or in matrix notation, $\varepsilon = A\beta$.

Writing $u_A(\beta) = u(A\beta) = u(\varepsilon)$, where $u(\varepsilon)$ is as above, we have, at $\beta = 0$,

$$\frac{\partial J_{i-1}(u_A(\beta))}{\partial \beta_j} = k_j^{*i} \qquad (i, j = 1, \ldots, N).$$

If we now define $F_i(z, t)$ as in Lemma 8.3 [**1**, p. 195], (with k_j^* instead of k_j), then the $F_i(z, t)$ will be continuous, and we may proceed as in the proof of Lemma 8.3. Thus, we conclude that there exist multipliers $\lambda_0 \geqslant 0$, $\lambda_1, \ldots, \lambda_{2n+1}$ such that $\sum_0^{2n+1} \lambda_i k^i \geqslant 0$ for all $k \in K$. In view of (7), this is exactly inequality (5).

In our specific case, $\lambda_0 \neq 0$. Indeed, suppose $\lambda_0 = 0$. Then we would have $F(t, u) = uT(t)$, where T is a trigonometric polynomial. If $T(t)$ were negative on some interval, then F could not have a minimum on that interval. If $T(t)$ were nonnegative, then the only $u_0(t) \geqslant 0$ which minimized F would be $u_0(t) \equiv 0$, but this u_0 does not satisfy the first side condition

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} u_0(t)\, dt = 1.$$

Thus, $\lambda_0 > 0$, and we may take $\lambda_0 = 1$.

REFERENCES

1. M. R. Hestenes, *Calculus of variations and optimal control theory*, Wiley, New York, 1966.

2. A. Pfluger, *Some coefficient problems of starlike functions*, Ann. Acad. Sci. Fenn. Ser. AI 2 (1976), 383–396.

DEPARTMENT OF MATHEMATICS, BAR-ILAN UNIVERSITY, RAMAT GAN, ISRAEL