MINIMAL \( H^2 \) INTERPOLATION IN
THE CARATHÉODORY CLASS

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Abstract. For \((c_1, \ldots, c_n)\) in \(C^n\), let \(C(c_1, \ldots, c_n)\) denote the class of
functions \(f(z) = 1 + c_1 z + \cdots + c_n z^n + \sum_{k=n+1}^\infty a_k z^k\) which are analytic
and satisfy \(\text{Re} f(z) > 0\) in the unit disc. The unique function of least \(H^2\)
norm in \(C(c_1, \ldots, c_n)\) is explicitly determined.

In this paper we consider a minimal interpolation problem at the origin, for
the class \(H^2 \cap C\), where \(H^2\) denotes the well-known Hardy space for the unit
disc, and \(C\) is the Carathéodory class of functions \(f(z) = 1 + c_1 z + c_2 z^2
+ \cdots\) which are analytic and satisfy \(\text{Re} f(z) > 0\) in \(|z| < 1\). In particular,
given \(n\) complex numbers \(c_1, \ldots, c_n\), we wish to find the function \(f\) in
\(H^2 \cap C\), of the form

\[
f(z) = 1 + c_1 z + \cdots + c_n z^n + \sum_{k=n+1}^\infty a_k z^k
\]

with least \(H^2\) norm.

For each \(n\), expansion (1) defines a mapping

\[
u_n: f \rightarrow (c_1, \ldots, c_n)
\]

of \(C\) onto some compact set \(C_n \subset C^n\), which is called the \(n\)th coefficient body
for \(C\). The following result is due to C. Carathéodory and O. Toeplitz (see
[2]):

**Theorem A.** \(C_n\) is a convex, compact body in \(C^n\). To each point in the interior
of \(C_n\) there correspond infinitely many functions in \(C\); but each boundary point of
\(C_n\) corresponds to only one \(f\) in \(C\). The boundary points correspond to functions
of the form

\[
f(z) = \sum_{k=1}^m \frac{1 + \alpha_k z}{1 - \alpha_k z} \mu_k,
\]

where \(1 < m < n\); \(|\alpha_k| = 1\), \(\mu_k > 0\) \((k = 1, \ldots, m)\), and \(\sum_{k=1}^m \mu_k = 1\).

From Theorem A it follows that a function corresponding to a boundary
point of \(C_n\) has infinite \(H^2\) norm. On the other hand, any interior point
corresponds to infinitely many functions in \(H^2 \cap C\). Indeed, if \((c_1, \ldots, c_n)\) is
interior to \(C_n\), choose \(0 < r < 1\) such that \((b_1, \ldots, b_n) \in C_n\), where \(b_k = \ldots\)

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Let \( g(z) \) be any function in \( C \) which corresponds to \( (b_1, \ldots, b_n) \). Then \( f(z) = g(rz) \) is still in \( C \), is now in \( H^2 \), and corresponds to \( (c_1, \ldots, c_n) \).

Thus, our interpolation problem is meaningful precisely for the interior of \( C_n \). If we let \( C(c_1, \ldots, c_n) \) denote the set of all \( f \) in \( C \) of the form (1), we have the following

**Theorem.** For each \( (c_1, \ldots, c_n) \) in the interior of \( C_n \), there exists a unique function \( f \) with least \( H^2 \) norm in \( C(c_1, \ldots, c_n) \). This \( f \) is of the form

\[
f(z) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{e^{it} + z}{e^{it} - z} u(t) \, dt,
\]

where \( u(t) = \max(0, P(t)) \), \( P(t) = a_0 + \sum_{k=1}^{\infty} (a_k \cos kt + b_k \sin kt) \), \( (a_k, b_k \) \text{ real}).

**Proof.** \( C(c_1, \ldots, c_n) \) is a normal and compact family, and \( ||f||_2 \) is a continuous functional. Thus, we are assured of the existence of an extremal \( f \).

It is classical that functions \( f \) in \( C \) have the Riesz-Herglotz representation

\[
f(z) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{e^{it} + z}{e^{it} - z} d\mu(t),
\]

where \( d\mu(t) \) is a positive measure on \([ -\pi, \pi] \) for which \( (1/2\pi)\int_{-\pi}^{\pi} d\mu(t) = 1 \). If we write

\[
\hat{\mu}(k) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-ikt} d\mu(t) \quad (k = 0, \pm 1, \pm 2, \ldots),
\]

then by comparing Taylor coefficients of both sides of (3), we obtain

\[
c_k = 2\hat{\mu}(k) \quad (k = 1, 2, \ldots) .
\]

We thus have a one-one correspondence between \( C(c_1, \ldots, c_n) \) and all positive measures \( d\mu(t) \) which satisfy

\[
\hat{\mu}(0) = 1, \quad \hat{\mu}(k) = \frac{1}{2} c_k \quad (k = 1, \ldots, n).
\]

Since

\[
||f||_2 = \left( 1 + 4\sum_{k=1}^{\infty} |\hat{\mu}(k)|^2 \right)^{1/2} ,
\]

and the \( L^2 \) norm of \( d\mu(t) \) is \( (1 + 2\sum_{k=1}^{\infty} |\hat{\mu}(k)|^2)^{1/2} \), it follows that \( f \in H^2 \cap C \) corresponds to \( u \in L^2[-\pi, \pi], u(t) \geq 0 \), where \( u(t)dt = d\mu(t) \).

Thus, we must find \( u_0 \in L^2[-\pi, \pi], u_0 \geq 0 \), which minimizes \( \int_{-\pi}^{\pi} F_j(t, u(t)) \, dt \) under the constraints

\[
\int_{-\pi}^{\pi} F_j(t, u(t)) \, dt = d_j \quad (j = 1, 2, \ldots, 2^{n+1}),
\]

where
\( F_0(u, t) = u^2, \)

\( F_{2k+1}(t, u) = u \cos kt \quad (k = 0, 1, \ldots, n), \)

\( F_{2k}(t, u) = u \sin kt \quad (k = 1, \ldots, n), \)

\( d_i = 2\pi; \quad d_{2k+1} + id_{2k} = \pi c_k \quad (k = 1, \ldots, n). \)

Of course, the existence of an extremal \( f \) implies that a corresponding extremal \( u_0 \) exists. We now need the following

**Lemma.** Suppose \( u_0 \) is the extremal function described above. Then there exist real multipliers \( \lambda_0 = 1, \lambda_1, \ldots, \lambda_{2n+1}, \) for which the function

\[
F = \sum_{k=0}^{2n+1} \lambda_k F_k
\]

is such that the inequality

\[
F(t, u) \geq F(t, u_0(t))
\]

holds for all pairs \((t, u) \in E \times [0, \infty), \) where \( E \subset [-\pi, \pi] \) contains almost all points of \([-\pi, \pi].\)

Assume for a moment that this Lemma has been established. Then by elementary calculus, the only function \( u_0(t) > 0 \) which minimizes \( F \) in the sense of (5) is

\[
u_0(t) = \max \left(0, -\frac{1}{2} \lambda_1 - \frac{1}{2} \sum_{k=1}^n \lambda_2 k \sin kt + \lambda_{2k+1} \cos kt \right) \text{ a.e.} \quad (6)
\]

For convenience, let \( u_0(t) \) denote the continuous representative of the \( L^2 \) equivalence class (6). Then we have the strict inequality \( F(t, u) > F(t, u_0(t)) \) for all \((t, u) \in [-\pi, \pi] \times [0, \infty), u \neq u_0(t). \) If we write

\[
J_k(u) = \int_{-\pi}^{\pi} F_k(t, u(t)) \, dt - d_k \quad (k = 0, 1, \ldots, 2n + 1; d_0 = 0),
\]

\[
J = \sum_{k=0}^{2n+1} \lambda_k j_k,
\]

then for any \( u(t) > 0 \) satisfying (4), we have \( J(u) = J(u_0). \) Thus, if such a \( u(t) \) differs from \( u_0(t) \) on a set of positive measure, we have \( J_0(u) = J(u) > J(u_0) = J_0(u_0). \) In other words, \( u_0 \) is the unique minimizing function. This in turn implies that the corresponding extremal \( f, \) given by (2), is unique. Thus, our Theorem is proven, subject to the Lemma.

**Proof of Lemma.** This Lemma is essentially an \( L^p \) version of Theorem 5.1 in [1, p. 215]. That variational theorem deals with piecewise continuous functions, but in our case, we do not know \textit{a priori} that the extremal \( u_0 \) is piecewise continuous. In the sequel, we follow the notation of Hestenes [1, Chapters 4, 5].

Let
\[ G_i(t) = \int_{-\pi}^{\pi} F_i(s, u_0(s)) ds \quad (i = 0, 1, \ldots, 2n + 1), \]

where \( u_0 \) is a nonnegative \( L^2 \) function which minimizes \( \int_{-\pi}^{\pi} F_0(t, u(t)) dt \) under the constraints (4). Let \( E \) be the set of all \( t \in (-\pi, \pi) \) such that \( G_i(t) = F_i(t, u_0(t)) \) \( (i = 0, 1, \ldots, 2n + 1) \). Note that \( [-\pi, \pi] - E \) has measure zero.

Let \( K \) be the class of all vectors \( k = (k^0, k^1, \ldots, k^{2n+1}) \) of the form

\[ k^i = F_i(t, u) - F_i(t, u_0(t)) \quad (i = 0, 1, \ldots, 2n + 1), \quad (7) \]

where \( t \in E, \ u > 0 \). For any finite set of vectors \( k_1, \ldots, k_M \) in \( K \), we define \( u(\epsilon) = u(\epsilon, t) \) as in [1, p. 221]. At the point \( \epsilon = 0 \), we have

\[ \frac{\partial J_\beta(u(\epsilon))}{\partial \epsilon_j} = k_j^i \quad (i = 0, 1, \ldots, 2n + 1; j = 1, \ldots, M). \]

Let \( K^* \) denote the cone generated by \( K \), and \( K^- \) the cone generated by the vector \((-1, 0, \ldots, 0)\). \( K^+ = K^* - K^- \) denotes the set of all vectors of the form \( k - k^- \), where \( k \in K^*, \ k^- \in K^- \). Set \( N = 2n + 2 \). Then Lemmas 8.1 and 8.2 follow as in [1, pp. 192–194]. What remains to be proven is that there exists a vector \( k \neq 0 \) which is not in the cone \( K^+ \).

Suppose that \( K^+ \) contained all vectors. Select \( N \) vectors \( k_1^*, \ldots, k_M^* \) in \( K^* \) having the properties described in Lemma 8.2. Since \( K^* \) is generated by \( K \), there exists a finite set of vectors \( k_1, \ldots, k_M \) in \( K \) such that

\[ k_j^* = \sum_{i=1}^{M} k_i a_{ij}, \quad a_{ij} > 0 \quad (j = 1, \ldots, N). \]

Writing \( \beta = (\beta_1, \ldots, \beta_N), (\beta_j > 0) \), we define \( e_i \) by

\[ e_i = \sum_{j=1}^{N} a_{ij} \beta_j \quad (i = 1, \ldots, M), \]

or in matrix notation, \( e = A\beta \).

Writing \( u_\lambda(\beta) = u(A\beta) = u(\epsilon) \), where \( u(\epsilon) \) is as above, we have, at \( \beta = 0, \)

\[ \frac{\partial J_{\lambda-1}(u_\lambda(\beta))}{\partial \beta_j} = k_j^* \quad (i, j = 1, \ldots, N). \]

If we now define \( F_i(z, t) \) as in Lemma 8.3 [1, p. 195], (with \( k_j^* \) instead of \( k_j \)), then the \( F_i(z, t) \) will be continuous, and we may proceed as in the proof of Lemma 8.3. Thus, we conclude that there exist multipliers \( \lambda_0 > 0, \lambda_1, \ldots, \lambda_{2n+1} \) such that \( \sum_{0}^{2n+1} \lambda_k i > 0 \) for all \( k \in K \). In view of (7), this is exactly inequality (5).

In our specific case, \( \lambda_0 \neq 0 \). Indeed, suppose \( \lambda_0 = 0 \). Then we would have \( F(t, u) = u T(t) \), where \( T \) is a trigonometric polynomial. If \( T(t) \) were negative on some interval, then \( F \) could not have a minimum on that interval. If \( T(t) \) were nonnegative, then the only \( u_0(t) > 0 \) which minimized \( F \) would be \( u_0(t) \equiv 0 \), but this \( u_0 \) does not satisfy the first side condition

\[ \frac{1}{2\pi} \int_{-\pi}^{\pi} u_0(t) dt = 1. \]
Thus, \( \lambda_0 > 0 \), and we may take \( \lambda_0 = 1 \).

REFERENCES


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