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INNER FUNCTIONS AND THE MAXIMAL IDEAL SPACE OF $H^\infty(U^n)$

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ABSTRACT. For the case of the polydisc, Range has shown that the Shilov boundary ∂_n of $H^\infty(U^n)$ is a proper subset of τX_n, the set of all restrictions of complex homomorphisms of $L^\infty(T^n)$ to $H^\infty(U^n)$. In this paper, we show that τX_n is a proper subset of those complex homomorphisms of $H^\infty(U^n)$ which are unimodular on the class of all inner functions.

1. Introduction. Let T^n be the distinguished boundary of the unit polydisc U^n and denote the class of all bounded analytic functions on U^n by $H^\infty(U^n)$. A function f in $H^\infty(U^n)$ is said to be inner if its radial boundary values

$$f^*(w) = \lim_{r\to 1} f(rw)$$

are of modulus one almost everywhere (a.e.) on T^n with respect to the normalised Lebesgue measure m_n on T^n. The class of all inner functions on U^n is denoted by Σ_n. Let

$$H^\infty/\Sigma_n = \{f^*/I^*: f \in H^\infty(U^n), I \in \Sigma_n\},$$

then its closure $[H^\infty/\Sigma_n]$ is the closed subalgebra generated by H^∞/Σ_n in $L^\infty(T^n)$. For the case of the unit disc U, the index $n = 1$ will be omitted from all our notations.

Douglas and Rudin [**1**] have shown that $[H^\infty/\Sigma_n] = L^\infty(T)$, the main result here shows that this is no longer true for $n > 1$.

2. A proper subalgebra of $L^\infty(T^n)$. First, a result of Rudin is needed, it will be stated with some additional details and estimates from the proof in [**3**, Theorem 5.4.8].

THEOREM 1. *Let A be a totally disconnected, compact subset of T with $m(A) > 0$. Defining $E_1 = \{(w_1, w_2) \in T^2: w_2/w_1 \in A\}$, E_1 is a compact, circular subset of T^2 with $m_2(E_1) > 0$. Then there exists F_1 in $H^\infty(U^2)$ such that*

(i) $3/5 > |F_1^*| > 2/5$ *a.e. on* E_1,
(ii) $11/10 > |F_1^*| > 9/10$ *a.e. on* $T^2 \setminus E_1$.
For $n \geqslant 2$, we define $F \in H^\infty(U^n)$ by

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$$F(z', z'') = F_1(z') \quad \textit{for } (z', z'') \in U^2 \times U^{n-2}.$$

Then

(i) $3/5 > |F^*| > 2/5$ *a.e. on* E,

(ii) $11/10 > |F^*| > 9/10$ *a.e. on* $T^n \setminus E$,

where $E = E_1 \times T^{n-2}$ *is a compact, circular subset of* T^n *with empty interior and* $m_n(E) > 0$.

It can now be shown that $[H^\infty/\Sigma_n]$ is a proper closed subalgebra of $L^\infty(T^n)$ for $n > 1$.

THEOREM 2. *There is an* F *in* $H^\infty(U^n)$ *such that* F^* *is invertible in* $L^\infty(T^n)$ *but is not invertible in* $[H^\infty/\Sigma_n]$ *for* $n > 1$.

PROOF. With F as defined after Theorem 1, F^* is clearly invertible in $L^\infty(T^n)$ as it is bounded away from 0. Now for any $f \in H^\infty(U^n)$, $I \in \Sigma_n$, suppose that

$$\left| \frac{1}{F^*} - \frac{f^*}{I^*} \right| < \frac{1}{9} \quad \text{a.e. on } T^n,$$

then

$$\left| \frac{1}{F^*} \right| - \frac{1}{9} < \left| \frac{f^*}{I^*} \right| = |f^*| < \left| \frac{1}{F^*} \right| + \frac{1}{9}.$$

Hence

$$|f^*| > 14/9 \quad \text{a.e. on } E, \qquad |f^*| < 11/9 \quad \text{a.e. on } T^n \setminus E.$$

Since E is circular,

$$G > 14/9 \quad \text{a.e. on } E, \qquad G < 11/9 \quad \text{a.e. on } T^n \setminus E,$$

where $G(w) = \operatorname{ess\,sup}_{|\alpha|=1} |f(\alpha w)|$. Suppose that $G = \psi$ almost everywhere and ψ is lower semicontinuous. Then

$$V = \{ w \in T^n : \psi(w) > 12/9 \}$$

is open and nonempty. But this is a contradiction as $G \neq \psi$ on $V \setminus E$ which is open and nonempty. Hence by a result of Rudin [**3**, Theorem 3.5.2], $f \notin H^\infty(U^n)$. This contradiction shows that

$$\operatorname{dist}\left(\frac{1}{F^*}, \frac{H^\infty}{\Sigma_n} \right) \geqslant \frac{1}{9} > 0.$$

Let M_n and X_n be the maximal ideal space of $H^\infty(U^n)$ and $L^\infty(T^n)$ respectively and define $\tau: X_n \to M_n$ by mapping each complex homomorphism of $L^\infty(T^n)$ to its restriction on $H^\infty(U^n)$. τX_n is then the image of the maximal ideal space of $L^\infty(T^n)$ in M_n. The proof of the lemma in [**1**, p. 317] can be used to show that for $n > 1$ too, the maximal ideal space $M[H^\infty/\Sigma_n]$ of $[H^\infty/\Sigma_n]$ can be identified with K_{Σ_n}, where

$$K_{\Sigma_n} = \{ \Phi \in M_n : |\Phi(I)| = 1 \text{ for all } I \in \Sigma_n \}.$$

Range [**2**] has shown that the Shilov boundary ∂_n of $H^\infty(U^n)$ is a proper subset of τX_n for $n > 1$. From Theorem 2, it can now be shown that again, unlike the case of the unit disc, $\tau X_n \neq K_{\Sigma_n}$.

THEOREM 3. *For* $n >,1$

$$\tau X_n \neq M[H^\infty/\Sigma_n] = K_{\Sigma_n}.$$

PROOF. With F as above, F^* generates a maximal ideal in $[H^\infty/\Sigma_n]$ as it is not invertible. The corresponding complex homomorphism cannot belong to τX_n since F^* is invertible in $L^\infty(T^n)$.

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