INNER FUNCTIONS AND THE MAXIMAL IDEAL SPACE OF $H^\infty(U^n)$

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ABSTRACT. For the case of the polydisc, Range has shown that the Shilov boundary $\partial_n$ of $H^\infty(U^n)$ is a proper subset of $\mathcal{R}_n$, the set of all restrictions of complex homomorphisms of $L^\infty(T^n)$ to $H^\infty(U^n)$. In this paper, we show that $\mathcal{R}_n$ is a proper subset of those complex homomorphisms of $H^\infty(U^n)$ which are unimodular on the class of all inner functions.

1. Introduction. Let $T^n$ be the distinguished boundary of the unit polydisc $U^n$ and denote the class of all bounded analytic functions on $U^n$ by $H^\infty(U^n)$. A function $f$ in $H^\infty(U^n)$ is said to be inner if its radial boundary values

$$f^\ast(w) = \lim_{r \to 1} f(rw)$$

are of modulus one almost everywhere (a.e.) on $T^n$ with respect to the normalised Lebesgue measure $m_n$ on $T^n$. The class of all inner functions on $U^n$ is denoted by $\mathcal{I}_n$. Let

$$H^\infty/\mathcal{I}_n = \{ f^\ast/\mathcal{I}^\ast : f \in H^\infty(U^n), \mathcal{I} \in \mathcal{I}_n \},$$

then its closure $[H^\infty/\mathcal{I}_n]$ is the closed subalgebra generated by $H^\infty/\mathcal{I}_n$ in $L^\infty(T^n)$. For the case of the unit disc $U$, the index $n = 1$ will be omitted from all our notations.

Douglas and Rudin [1] have shown that $[H^\infty/\mathcal{I}_n] = L^\infty(T)$, the main result here shows that this is no longer true for $n > 1$.

2. A proper subalgebra of $L^\infty(T^n)$. First, a result of Rudin is needed, it will be stated with some additional details and estimates from the proof in [3, Theorem 5.4.8].

Theorem 1. Let $A$ be a totally disconnected, compact subset of $T$ with $m(A) > 0$. Defining $E_1 = \{(w_1, w_2) \in T^2 : w_2/w_1 \in A\}$, $E_1$ is a compact, circular subset of $T^2$ with $m_2(E_1) > 0$. Then there exists $F_1$ in $H^\infty(U^2)$ such that

(i) $3/5 > |F_1^\ast| > 2/5$ a.e. on $E_1$,
(ii) $11/10 > |F_1^\ast| > 9/10$ a.e. on $T^2 \setminus E_1$.

For $n \geq 2$, we define $F \in H^\infty(U^n)$ by

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$F(z', z'') = F_1(z')$ for $(z', z'') \in U^2 \times U^{n-2}$.

Then
(i) $3/5 > |F^*| > 2/5$ a.e. on $E$,
(ii) $11/10 > |F^*| > 9/10$ a.e. on $T^n \setminus E$,

where $E = E_1 \times T^{n-2}$ is a compact, circular subset of $T^n$ with empty interior and $m_n(E) > 0$.

It can now be shown that $[H^\infty/\Sigma_n]$ is a proper closed subalgebra of $L^\infty(T^n)$ for $n > 1$.

**Theorem 2.** There is an $F$ in $H^\infty(U^n)$ such that $F^*$ is invertible in $L^\infty(T^n)$ but is not invertible in $[H^\infty/\Sigma_n]$ for $n > 1$.

**Proof.** With $F$ as defined after Theorem 1, $F^*$ is clearly invertible in $L^\infty(T^n)$ as it is bounded away from 0. Now for any $f \in H^\infty(U^n)$, $I \in \Sigma_n$, suppose that
\[
\left| \frac{1}{F^*} - \frac{f^*}{I^*} \right| < \frac{1}{9} \quad \text{a.e. on } T^n,
\]

then
\[
\left| \frac{1}{F^*} - \frac{1}{9} \right| < \left| \frac{f^*}{I^*} \right| = \left| f^* \right| < \left| \frac{1}{F^*} \right| + \frac{1}{9}.
\]

Hence
\[
\left| f^* \right| > \frac{14}{9} \quad \text{a.e. on } E, \quad \left| f^* \right| < \frac{11}{9} \quad \text{a.e. on } T^n \setminus E.
\]

Since $E$ is circular,
\[
G > \frac{14}{9} \quad \text{a.e. on } E, \quad G < \frac{11}{9} \quad \text{a.e. on } T^n \setminus E,
\]

where $G(w) = \text{ess sup}_{|a| = 1} |f(aw)|$. Suppose that $G = \psi$ almost everywhere and $\psi$ is lower semicontinuous. Then
\[
V = \{ w \in T^n : \psi(w) > 12/9 \}
\]
is open and nonempty. But this is a contradiction as $G \neq \psi$ on $V \setminus E$ which is open and nonempty. Hence by a result of Rudin [3, Theorem 3.5.2], $f \notin H^\infty(U^n)$. This contradiction shows that
\[
\text{dist}\left(\frac{1}{F^*}, \frac{H^\infty}{\Sigma_n}\right) > \frac{1}{9} > 0.
\]

Let $M_n$ and $X_n$ be the maximal ideal space of $H^\infty(U^n)$ and $L^\infty(T^n)$ respectively and define $\tau : X_n \to M_n$ by mapping each complex homomorphism of $L^\infty(T^n)$ to its restriction on $H^\infty(U^n)$. $\tau X_n$ is then the image of the maximal ideal space of $L^\infty(T^n)$ in $M_n$. The proof of the lemma in [1, p. 317] can be used to show that for $n > 1$ too, the maximal ideal space $M[H^\infty/\Sigma_n]$ of $[H^\infty/\Sigma_n]$ can be identified with $K_{\Sigma_n}$, where
\[
K_{\Sigma_n} = \{ \Phi \in M_n : |\Phi(I)| = 1 \text{ for all } I \in \Sigma_n \}.
\]

Range [2] has shown that the Shilov boundary $\delta_n$ of $H^\infty(U^n)$ is a proper
subset of $\tau X_n$ for $n > 1$. From Theorem 2, it can now be shown that again, unlike the case of the unit disc, $\tau X_n \neq K_{\Sigma_n}$.

**Theorem 3.** For $n > 1$

$$\tau X_n \neq M[ H^\infty / \Sigma_n ] = K_{\Sigma_n}.$$ 

**Proof.** With $F$ as above, $F^*$ generates a maximal ideal in $[ H^\infty / \Sigma_n ]$ as it is not invertible. The corresponding complex homomorphism cannot belong to $\tau X_n$ since $F^*$ is invertible in $L^\infty (T^n)$.

**References**


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