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ON THE STABILITY OF THE LINEAR MAPPING IN BANACH SPACES

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ABSTRACT. Let E_1, E_2 be two Banach spaces, and let $f\colon E_1 \to E_2$ be a mapping, that is "approximately linear". S. M. Ulam posed the problem: "Give conditions in order for a linear mapping near an approximately linear mapping to exist". The purpose of this paper is to give an answer to Ulam's problem.

THEOREM. *Consider E_1, E_2 to be two Banach spaces, and let $f\colon E_1 \to E_2$ be a mapping such that $f(tx)$ is continuous in t for each fixed x. Assume that there exists $\theta \geqslant 0$ and $p \in [0, 1)$ such that*

$$\frac{\|f(x+y) - f(x) - f(y)\|}{\|x\|^p + \|y\|^p} \leqslant \theta, \quad \text{for any } x, y \in E_1. \tag{1}$$

Then there exists a unique linear mapping $T\colon E_1 \to E_2$ such that

$$\frac{\|f(x) - T(x)\|}{\|x\|^p} \leqslant \frac{2\theta}{2 - 2^p}, \quad \text{for any } x \in E_1. \tag{2}$$

PROOF. Claim that

$$\frac{\|[f(2^n x)]/2^n - f(x)\|}{\|x\|^p} \leqslant \theta \sum_{m=0}^{n-1} 2^{m(p-1)} \tag{3}$$

for any integer n, and some $\theta \geqslant 0$. The verification of (3) follows by induction on n. Indeed the case $n = 1$ is clear because by the hypothesis we can find θ, that is greater or equal to zero, and p such that $0 \leqslant p < 1$ with

$$\frac{\|[f(2x)]/2 - f(x)\|}{\|x\|^p} \leqslant \theta. \tag{4}$$

Assume now that (3) holds and we want to prove it for the case $(n + 1)$. However this is true because by (3) we obtain

$$\frac{\|[f(2^n \cdot 2x)]/2^n - f(2x)\|}{\|2x\|^p} \leqslant \theta \sum_{m=0}^{n-1} 2^{m(p-1)},$$

therefore

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$$\frac{\|[f(2^{n+1}x)]/2^{n+1} - \frac{1}{2}f(2x)\|}{\|x\|^p} \leqslant \theta \sum_{m=1}^{n} 2^{m(p-1)}.$$

By the triangle inequality we obtain

$$\left\| \frac{1}{2^{n+1}}\left[f(2^{n+1}x)\right] - f(x) \right\| \leqslant \left\| \frac{1}{2^{n+1}}\left[f(2^{n+1}x)\right] - \frac{1}{2}\left[f(2x)\right] \right\| + \left\| \frac{1}{2}\left[f(2x)\right] - f(x) \right\| \leqslant \theta\|x\|^p \sum_{m=0}^{n} 2^{m(p-1)}.$$

Thus

$$\frac{\|[f(2^{n+1}x)]/2^{n+1} - f(x)\|}{\|x\|^p} \leqslant \theta \sum_{m=0}^{n} 2^{m(p-1)}$$

and (3) is valid for any integer n. It follows then that

$$\frac{\|[f(2^n x)]/2^n - f(x)\|}{\|x\|^p} \leqslant \frac{2\theta}{2 - 2^p}, \tag{5}$$

because $\sum_{m=0}^{\infty} 2^{m(p-1)}$ converges to $2/(2 - 2^p)$, as $0 \leqslant p < 1$. However, for $m > n > 0$,

$$\left\| \frac{1}{2^m}\left[f(2^m x)\right] - \frac{1}{2^n}\left[f(2^n x)\right] \right\| = \frac{1}{2^n} \left\| \frac{1}{2^{m-n}}\left[f(2^m x)\right] - \left[f(2^n x)\right] \right\| < 2^{n(p-1)} \cdot \frac{2\theta}{2 - 2^p} \|x\|^p.$$

Therefore

$$\lim_{n\to\infty} \left\| \frac{1}{2^m}\left[f(2^m x)\right] - \frac{1}{2^n}\left[f(2^n x)\right] \right\| = 0.$$

But E_2, as a Banach space, is complete, thus the sequence $\{[f(2^n x)]/2^n\}$ converges. Set

$$T(x) \equiv \lim_{n\to\infty} \frac{1}{2^n}\left[f(2^n x)\right].$$

It follows that

$$\|f[2^n(x+y)] - f[2^n x] - f[2^n y]\| \leqslant \theta(\|2^n x\|^p + \|2^n y\|^p) = 2^{np}\theta(\|x\|^p + \|y\|^p).$$

Therefore

$$\frac{1}{2^n} \|f[2^n(x+y)] - f[2^n x] - f[2^n y]\| \leqslant 2^{n(p-1)} \cdot \theta(\|x\|^p + \|y\|^p)$$

or

$$\lim_{n\to\infty} \frac{1}{2^n} \|f[2^n(x+y)] - f[2^n x] - f[2^n y]\| \leqslant \lim_{n\to\infty} 2^{n(p-1)}\theta(\|x\|^p + \|y\|^p)$$

or

$$\left\| \lim_{n\to\infty} \frac{1}{2^n} f[2^n(x+y)] - \lim_{n\to\infty} \frac{1}{2^n} f[2^n x] - \lim_{n\to\infty} \frac{1}{2^n} f[2^n y] \right\| = 0$$

or

$$\|T(x+y) - T(x) - T(y)\| = 0 \quad \text{for any } x, y \in E_1$$

or

$$T(x+y) = T(x) + T(y) \quad \text{for all } x, y \in E_1.$$

Since $T(x+y) = T(x) + T(y)$ for any $x, y \in E_1$, $T(rx) = rT(x)$ for any rational number r. Fix $x_0 \in E_1$ and $\rho \in E_2^*$ (the dual space of E_2). Consider the mapping

$$\mathbf{R} \ni t \mapsto \rho(T(tx)) = \phi(t).$$

Then ϕ: $\mathbf{R} \to \mathbf{R}$ satisfies the property that $\phi(a+b) = \phi(a) + \phi(b)$, i.e. ϕ is a group homomorphism. Moreover ϕ is a Borel function, because of the following reasoning. Let $\phi(t) = \lim_{n\to\infty} \rho(f(2^n tx_0))/2^n$ and set $\phi_n(t) = \rho(f(2^n tx_0))/2^n$. Then $\phi_n(t)$ are continuous functions. But $\phi(t)$ is the pointwise limit of continuous functions, thus $\phi(t)$ is a Borel function. It is a known fact that if ϕ: $\mathbf{R}^n \to \mathbf{R}^n$ is a function such that ϕ is a group homomorphism, i.e. $\phi(x+y) = \phi(x) + \phi(y)$ and ϕ is a measurable function, then ϕ is continuous. In fact this statement is also true if we replace $\mathbf{R}^n$ by any separable, locally compact abelian group (see for example: W. Rudin [3]). Therefore $\phi(t)$ is a continuous function. Let $a \in \mathbf{R}$. Then $a = \lim_{n\to\infty} r_n$, where $\{r_n\}$ is a sequence of rational numbers. Hence

$$\phi(at) = \phi\left(t \lim_{n\to\infty} r_n\right) = \lim_{n\to\infty} \phi(tr_n) = \left(\lim_{n\to\infty} r_n\right)\phi(t) = a\phi(t).$$

Therefore $\phi(at) = a\phi(t)$ for any $a \in \mathbf{R}$. Thus $T(ax) = aT(x)$ for any $a \in \mathbf{R}$. Hence T is a linear mapping.

From (5) we obtain

$$\lim_{n\to\infty} \frac{\|[f(2^n x)]/2^n - f(x)\|}{\|x\|^p} \leqslant \lim_{n\to\infty} \frac{2\theta}{2 - 2^p}$$

or equivalently,

$$\frac{\|T(x) - f(x)\|}{\|x\|^p} \leqslant \varepsilon, \quad \text{where } \varepsilon = \frac{2\theta}{2 - 2^p}, \tag{6}$$

Thus we have obtained (2).

We want now to prove that T is the unique such linear mapping. Assume that there exists another one, denoted by g: $E_1 \to E_2$ such that $T(x) \not\equiv g(x)$, $x \in E_1$. Then there exists a constant ε_1, greater or equal to zero, and q such that $0 \leqslant q < 1$ with

$$\frac{\|g(x) - f(x)\|}{\|x\|^q} \leqslant \varepsilon_1. \tag{7}$$

By the triangle inequality and (6) we obtain

$$\|T(x) - g(x)\| \leqslant \|T(x) - f(x)\| + \|f(x) - g(x)\| \leqslant \varepsilon\|x\|^p + \varepsilon_1\|x\|^q.$$

Therefore

$$\|T(x) - g(x)\| = \left\| \frac{1}{n}\left[T(nx)\right] - \frac{1}{n}\left[g(nx)\right] \right\| = \frac{1}{n}\|T(nx) - g(nx)\|$$
$$\leqslant \frac{1}{n}\left(\varepsilon\|nx\|^p + \varepsilon_1\|nx\|^q\right) = n^{p-1}\varepsilon\|x\|^p + n^{q-1}\varepsilon_1\|x\|^q.$$

Thus $\lim_{n\to\infty}\|T(x) - g(x)\| = 0$ for all $x \in E_1$ and hence $T(x) \equiv g(x)$ for all $x \in E_1$. Q.E.D.

This solves a problem posed by S. M. Ulam **[4]**, **[5]**: *When does a linear mapping near an "approximately linear" mapping exist*? The case $p = 0$ was answered by D. H. Hyers **[1]**. Thus we have succeeded here to give a generalized solution to Ulam's problem.

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