ON THE STABILITY OF THE LINEAR MAPPING IN BANACH SPACES

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Abstract. Let $E_1, E_2$ be two Banach spaces, and let $f: E_1 \to E_2$ be a mapping, that is "approximately linear". S. M. Ulam posed the problem: "Give conditions in order for a linear mapping near an approximately linear mapping to exist". The purpose of this paper is to give an answer to Ulam's problem.

Theorem. Consider $E_1, E_2$ to be two Banach spaces, and let $f: E_1 \to E_2$ be a mapping such that $f(tx)$ is continuous in $t$ for each fixed $x$. Assume that there exists $\theta > 0$ and $p \in [0, 1)$ such that
\[
\frac{\|f(x+y) - f(x) - f(y)\|}{\|x\|^p + \|y\|^p} \leq \theta, \quad \text{for any } x, y \in E_1.
\]
Then there exists a unique linear mapping $T: E_1 \to E_2$ such that
\[
\frac{\|f(x) - T(x)\|}{\|x\|^p} \leq \frac{2\theta}{2 - 2^p}, \quad \text{for any } x \in E_1.
\]

Proof. Claim that
\[
\frac{\|f(2^n x) - f(x)\|}{\|x\|^p} \leq \theta \sum_{m=0}^{n-1} 2^{m(p-1)}
\]
for any integer $n$, and some $\theta > 0$. The verification of (3) follows by induction on $n$. Indeed the case $n = 1$ is clear because by the hypothesis we can find $\theta$, that is greater or equal to zero, and $p$ such that $0 < p < 1$ with
\[
\frac{\|f(2x) - f(x)\|}{\|x\|^p} \leq \theta.
\]
Assume now that (3) holds and we want to prove it for the case $(n + 1)$. However this is true because by (3) we obtain
\[
\frac{\|f(2^n \cdot 2x) - f(2x)\|}{\|2x\|^p} \leq \theta \sum_{m=0}^{n-1} 2^{m(p-1)},
\]
therefore
By the triangle inequality we obtain
\[
\left\| \frac{1}{2^n + 1} \left[ f(2^n + 1)x \right] - f(x) \right\| < \frac{1}{2^n + 1} \left\| f(2^n + 1)x \right\| - \frac{1}{2} \left[ f(2^n + 1)x \right] - \frac{1}{2} \left[ f(2^n) - f(x) \right] \leq \theta \|x\|^p \sum_{m=0}^{n} 2^{m(p-1)}.
\]

Thus
\[
\frac{\left\| \frac{1}{2^n + 1} \left[ f(2^n + 1)x \right] - f(x) \right\|}{\|x\|^p} < \theta \sum_{m=0}^{n} 2^{m(p-1)}
\]
and (3) is valid for any integer \(n\). It follows then that
\[
\frac{\left\| \frac{1}{2^n} \left[ f(2^n x) \right] - f(x) \right\|}{\|x\|^p} \leq \frac{2\theta}{2 - 2^p}, \tag{5}
\]
because \(\sum_{m=0}^{\infty} 2^{m(p-1)}\) converges to \(2/(2 - 2^p)\), as \(0 < p < 1\). However, for \(m > n > 0\),
\[
\left\| \frac{1}{2^m} \left[ f(2^m x) \right] - \frac{1}{2^n} \left[ f(2^n x) \right] \right\| = \frac{1}{2^n} \left\| \frac{1}{2^{m-n}} \left[ f(2^n x) \right] - \left[ f(2^m x) \right] \right\| \leq 2^m(2^{m-n}) - \frac{2\theta}{2 - 2^p} \|x\|^p.
\]
Therefore
\[
\lim_{n \to \infty} \left\| \frac{1}{2^m} \left[ f(2^m x) \right] - \frac{1}{2^n} \left[ f(2^n x) \right] \right\| = 0.
\]
But \(E_2\), as a Banach space, is complete, thus the sequence \(\{f(2^n x)/2^n\}\) converges. Set
\[
T(x) \equiv \lim_{n \to \infty} \frac{1}{2^n} \left[ f(2^n x) \right].
\]
It follows that
\[
\left\| f[2^n(x + y)] - f[2^n x] - f[2^n y] \right\| \leq \theta (\|x\|^p + \|y\|^p) = 2^p \theta (\|x\|^p + \|y\|^p).
\]
Therefore
\[
\frac{1}{2^n} \left\| f[2^n(x + y)] - f[2^n x] - f[2^n y] \right\| \leq 2^{n(p-1)} \theta (\|x\|^p + \|y\|^p)
\]
or
\[
\lim_{n \to \infty} \frac{1}{2^n} \left\| f[2^n(x + y)] - f[2^n x] - f[2^n y] \right\| \leq \lim_{n \to \infty} 2^{n(p-1)} \theta (\|x\|^p + \|y\|^p)
\]
or
\[ \lim_{n \to \infty} \frac{1}{2^n} f[2^n(x + y)] - \lim_{n \to \infty} \frac{1}{2^n} f[2^n x] - \lim_{n \to \infty} \frac{1}{2^n} f[2^n y] = 0 \]

or

\[ \|T(x + y) - T(x) - T(y)\| = 0 \quad \text{for any } x, y \in E_1 \]

or

\[ T(x + y) = T(x) + T(y) \quad \text{for all } x, y \in E_1. \]

Since \( T(x + y) = T(x) + T(y) \) for any \( x, y \in E_1 \), \( T(rx) = rT(x) \) for any rational number \( r \). Fix \( x_0 \in E_1 \) and \( \rho \in E_2^* \) (the dual space of \( E_2 \)). Consider the mapping

\[ \mathbb{R} \ni t \mapsto \rho(T(tx)) = \phi(t). \]

Then \( \phi: \mathbb{R} \to \mathbb{R} \) satisfies the property that \( \phi(a + b) = \phi(a) + \phi(b) \), i.e. \( \phi \) is a group homomorphism. Moreover \( \phi \) is a Borel function, because of the following reasoning. Let \( \phi(t) = \lim_{n \to \infty} \rho(f(2^n tx_0))/2^n \) and set \( \phi_n(t) = \rho(f(2^n tx_0))/2^n \). Then \( \phi_n(t) \) are continuous functions. But \( \phi(t) \) is the pointwise limit of continuous functions, thus \( \phi(t) \) is a Borel function. It is a known fact that if \( \phi: \mathbb{R}^n \to \mathbb{R}^n \) is a function such that \( \phi \) is a group homomorphism, i.e. \( \phi(x + y) = \phi(x) + \phi(y) \) and \( \phi \) is a measurable function, then \( \phi \) is continuous. In fact this statement is also true if we replace \( \mathbb{R}^n \) by any separable, locally compact abelian group (see for example: W. Rudin [3]).

Therefore \( \phi(t) \) is a continuous function. Let \( a \in \mathbb{R} \). Then \( a = \lim_{n \to \infty} r_n \), where \( \{r_n\} \) is a sequence of rational numbers. Hence

\[ \phi(at) = \phi \left( \lim_{n \to \infty} r_n \right) = \lim_{n \to \infty} \phi(tr_n) = \left( \lim_{n \to \infty} r_n \right) \phi(t) = a\phi(t). \]

Therefore \( \phi(at) = a\phi(t) \) for any \( a \in \mathbb{R} \). Thus \( T(ax) = aT(x) \) for any \( a \in \mathbb{R} \).

Hence \( T \) is a linear mapping.

From (5) we obtain

\[ \lim_{n \to \infty} \frac{\left\| f(2^n x) \right\|/2^n - f(x)}{\|x\|^p} \leq \lim_{n \to \infty} \frac{2\theta}{2 - 2^p} \]

or equivalently,

\[ \frac{\|T(x) - f(x)\|}{\|x\|^p} \leq \varepsilon, \quad \text{where } \varepsilon = \frac{2\theta}{2 - 2^p}, \tag{6} \]

Thus we have obtained (2).

We want now to prove that \( T \) is the unique such linear mapping. Assume that there exists another one, denoted by \( g: E_1 \to E_2 \) such that \( T(x) \not\equiv g(x) \), \( x \in E_1 \). Then there exists a constant \( \varepsilon_1 \), greater or equal to zero, and \( q \) such that \( 0 < q < 1 \) with

\[ \frac{\|g(x) - f(x)\|}{\|x\|^q} \leq \varepsilon_1. \tag{7} \]

By the triangle inequality and (6) we obtain
\[ \| T(x) - g(x) \| \leq \| T(x) - f(x) \| + \| f(x) - g(x) \| \leq \varepsilon \| x \|^p + \varepsilon_1 \| x \|^q. \]

Therefore

\[ \| T(x) - g(x) \| = \left\| \frac{1}{n} \left[ T(nx) - g(nx) \right] \right\| = \frac{1}{n} \| T(nx) - g(nx) \| \]

\[ \leq \frac{1}{n} \left( \varepsilon \| nx \|^p + \varepsilon_1 \| nx \|^q \right) = n^{p-1} \varepsilon \| x \|^p + n^{q-1} \varepsilon_1 \| x \|^q. \]

Thus \( \lim_{n \to \infty} \| T(x) - g(x) \| = 0 \) for all \( x \in E_1 \) and hence \( T(x) \equiv g(x) \) for all \( x \in E_1 \). Q.E.D.

This solves a problem posed by S. M. Ulam [4], [5]: When does a linear mapping near an "approximately linear" mapping exist? The case \( p = 0 \) was answered by D. H. Hyers [1]. Thus we have succeeded here to give a generalized solution to Ulam’s problem.

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REFERENCES


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