

## ON THE STABILITY OF THE LINEAR MAPPING IN BANACH SPACES

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**ABSTRACT.** Let  $E_1, E_2$  be two Banach spaces, and let  $f: E_1 \rightarrow E_2$  be a mapping, that is "approximately linear". S. M. Ulam posed the problem: "Give conditions in order for a linear mapping near an approximately linear mapping to exist". The purpose of this paper is to give an answer to Ulam's problem.

**THEOREM.** Consider  $E_1, E_2$  to be two Banach spaces, and let  $f: E_1 \rightarrow E_2$  be a mapping such that  $f(tx)$  is continuous in  $t$  for each fixed  $x$ . Assume that there exists  $\theta \geq 0$  and  $p \in [0, 1)$  such that

$$\frac{\|f(x+y) - f(x) - f(y)\|}{\|x\|^p + \|y\|^p} \leq \theta, \quad \text{for any } x, y \in E_1. \quad (1)$$

Then there exists a unique linear mapping  $T: E_1 \rightarrow E_2$  such that

$$\frac{\|f(x) - T(x)\|}{\|x\|^p} \leq \frac{2\theta}{2 - 2^p}, \quad \text{for any } x \in E_1. \quad (2)$$

**PROOF.** Claim that

$$\frac{\|[f(2^n x)]/2^n - f(x)\|}{\|x\|^p} \leq \theta \sum_{m=0}^{n-1} 2^{m(p-1)} \quad (3)$$

for any integer  $n$ , and some  $\theta \geq 0$ . The verification of (3) follows by induction on  $n$ . Indeed the case  $n = 1$  is clear because by the hypothesis we can find  $\theta$ , that is greater or equal to zero, and  $p$  such that  $0 < p < 1$  with

$$\frac{\|[f(2x)]/2 - f(x)\|}{\|x\|^p} \leq \theta. \quad (4)$$

Assume now that (3) holds and we want to prove it for the case  $(n + 1)$ . However this is true because by (3) we obtain

$$\frac{\|[f(2^n \cdot 2x)]/2^n - f(2x)\|}{\|2x\|^p} \leq \theta \sum_{m=0}^{n-1} 2^{m(p-1)},$$

therefore

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$$\frac{\| [f(2^{n+1}x)]/2^{n+1} - \frac{1}{2}f(2x) \|}{\|x\|^p} < \theta \sum_{m=1}^n 2^{m(p-1)}.$$

By the triangle inequality we obtain

$$\begin{aligned} \left\| \frac{1}{2^{n+1}} [f(2^{n+1}x)] - f(x) \right\| &\leq \left\| \frac{1}{2^{n+1}} [f(2^{n+1}x)] - \frac{1}{2} [f(2x)] \right\| \\ &\quad + \left\| \frac{1}{2} [f(2x)] - f(x) \right\| \leq \theta \|x\|^p \sum_{m=0}^n 2^{m(p-1)}. \end{aligned}$$

Thus

$$\frac{\| [f(2^{n+1}x)]/2^{n+1} - f(x) \|}{\|x\|^p} \leq \theta \sum_{m=0}^n 2^{m(p-1)}$$

and (3) is valid for any integer  $n$ . It follows then that

$$\frac{\| [f(2^n x)]/2^n - f(x) \|}{\|x\|^p} \leq \frac{2\theta}{2-2^p}, \quad (5)$$

because  $\sum_{m=0}^{\infty} 2^{m(p-1)}$  converges to  $2/(2-2^p)$ , as  $0 < p < 1$ . However, for  $m > n > 0$ ,

$$\begin{aligned} \left\| \frac{1}{2^m} [f(2^m x)] - \frac{1}{2^n} [f(2^n x)] \right\| &= \frac{1}{2^n} \left\| \frac{1}{2^{m-n}} [f(2^m x)] - [f(2^n x)] \right\| \\ &< 2^{n(p-1)} \cdot \frac{2\theta}{2-2^p} \|x\|^p. \end{aligned}$$

Therefore

$$\lim_{n \rightarrow \infty} \left\| \frac{1}{2^n} [f(2^n x)] - \frac{1}{2^n} [f(2^n x)] \right\| = 0.$$

But  $E_2$ , as a Banach space, is complete, thus the sequence  $\{[f(2^n x)]/2^n\}$  converges. Set

$$T(x) \equiv \lim_{n \rightarrow \infty} \frac{1}{2^n} [f(2^n x)].$$

It follows that

$$\begin{aligned} \|f[2^n(x+y)] - f[2^n x] - f[2^n y]\| &\leq \theta (\|2^n x\|^p + \|2^n y\|^p) \\ &= 2^{np} \theta (\|x\|^p + \|y\|^p). \end{aligned}$$

Therefore

$$\frac{1}{2^n} \|f[2^n(x+y)] - f[2^n x] - f[2^n y]\| \leq 2^{n(p-1)} \cdot \theta (\|x\|^p + \|y\|^p)$$

or

$$\lim_{n \rightarrow \infty} \frac{1}{2^n} \|f[2^n(x+y)] - f[2^n x] - f[2^n y]\| \leq \lim_{n \rightarrow \infty} 2^{n(p-1)} \theta (\|x\|^p + \|y\|^p)$$

or

$$\left\| \lim_{n \rightarrow \infty} \frac{1}{2^n} f[2^n(x+y)] - \lim_{n \rightarrow \infty} \frac{1}{2^n} f[2^n x] - \lim_{n \rightarrow \infty} \frac{1}{2^n} f[2^n y] \right\| = 0$$

or

$$\|T(x+y) - T(x) - T(y)\| = 0 \quad \text{for any } x, y \in E_1$$

or

$$T(x+y) = T(x) + T(y) \quad \text{for all } x, y \in E_1.$$

Since  $T(x+y) = T(x) + T(y)$  for any  $x, y \in E_1$ ,  $T(rx) = rT(x)$  for any rational number  $r$ . Fix  $x_0 \in E_1$  and  $\rho \in E_2^*$  (the dual space of  $E_2$ ). Consider the mapping

$$\mathbf{R} \ni t \mapsto \rho(T(tx)) = \phi(t).$$

Then  $\phi: \mathbf{R} \rightarrow \mathbf{R}$  satisfies the property that  $\phi(a+b) = \phi(a) + \phi(b)$ , i.e.  $\phi$  is a group homomorphism. Moreover  $\phi$  is a Borel function, because of the following reasoning. Let  $\phi(t) = \lim_{n \rightarrow \infty} \rho(f(2^n tx_0))/2^n$  and set  $\phi_n(t) = \rho(f(2^n tx_0))/2^n$ . Then  $\phi_n(t)$  are continuous functions. But  $\phi(t)$  is the pointwise limit of continuous functions, thus  $\phi(t)$  is a Borel function. It is a known fact that if  $\phi: \mathbf{R}^n \rightarrow \mathbf{R}^n$  is a function such that  $\phi$  is a group homomorphism, i.e.  $\phi(x+y) = \phi(x) + \phi(y)$  and  $\phi$  is a measurable function, then  $\phi$  is continuous. In fact this statement is also true if we replace  $\mathbf{R}^n$  by any separable, locally compact abelian group (see for example: W. Rudin [3]). Therefore  $\phi(t)$  is a continuous function. Let  $a \in \mathbf{R}$ . Then  $a = \lim_{n \rightarrow \infty} r_n$ , where  $\{r_n\}$  is a sequence of rational numbers. Hence

$$\phi(at) = \phi\left(t \lim_{n \rightarrow \infty} r_n\right) = \lim_{n \rightarrow \infty} \phi(tr_n) = \left(\lim_{n \rightarrow \infty} r_n\right)\phi(t) = a\phi(t).$$

Therefore  $\phi(at) = a\phi(t)$  for any  $a \in \mathbf{R}$ . Thus  $T(ax) = aT(x)$  for any  $a \in \mathbf{R}$ . Hence  $T$  is a linear mapping.

From (5) we obtain

$$\lim_{n \rightarrow \infty} \frac{\| [f(2^n x)]/2^n - f(x) \|}{\|x\|^p} \leq \lim_{n \rightarrow \infty} \frac{2\theta}{2 - 2^p}$$

or equivalently,

$$\frac{\|T(x) - f(x)\|}{\|x\|^p} \leq \varepsilon, \quad \text{where } \varepsilon = \frac{2\theta}{2 - 2^p}, \tag{6}$$

Thus we have obtained (2).

We want now to prove that  $T$  is the unique such linear mapping. Assume that there exists another one, denoted by  $g: E_1 \rightarrow E_2$  such that  $T(x) \cong g(x)$ ,  $x \in E_1$ . Then there exists a constant  $\varepsilon_1$ , greater or equal to zero, and  $q$  such that  $0 \leq q < 1$  with

$$\frac{\|g(x) - f(x)\|}{\|x\|^q} \leq \varepsilon_1. \tag{7}$$

By the triangle inequality and (6) we obtain

$$\|T(x) - g(x)\| \leq \|T(x) - f(x)\| + \|f(x) - g(x)\| \leq \varepsilon \|x\|^p + \varepsilon_1 \|x\|^q.$$

Therefore

$$\begin{aligned} \|T(x) - g(x)\| &= \left\| \frac{1}{n} [T(nx)] - \frac{1}{n} [g(nx)] \right\| = \frac{1}{n} \|T(nx) - g(nx)\| \\ &\leq \frac{1}{n} (\varepsilon \|nx\|^p + \varepsilon_1 \|nx\|^q) = n^{p-1} \varepsilon \|x\|^p + n^{q-1} \varepsilon_1 \|x\|^q. \end{aligned}$$

Thus  $\lim_{n \rightarrow \infty} \|T(x) - g(x)\| = 0$  for all  $x \in E_1$  and hence  $T(x) \equiv g(x)$  for all  $x \in E_1$ . Q.E.D.

This solves a problem posed by S. M. Ulam [4], [5]: *When does a linear mapping near an "approximately linear" mapping exist?* The case  $p = 0$  was answered by D. H. Hyers [1]. Thus we have succeeded here to give a generalized solution to Ulam's problem.

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