EXTREME INVARIANT POSITIVE OPERATORS ON $L_p$-SPACES

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Abstract. Let $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{B}, \nu)$ be finite positive measure spaces. In this note we present characterizations of the extreme points of the convex set of all positive linear operators $T: L_p(\mu) \to L_q(\nu)$ with $T 1_X = 1_Y$ which are invariant with respect to a semigroup of positive constant preserving contractions on $L_p(\mu)$, $1 < p < \infty$, $1 < q < \infty$.

Introduction. Let $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{B}, \nu)$ be finite positive measure spaces and let $K[L_p(\mu), L_q(\nu)]$ denote the convex set of all positive linear operators $T: L_p(\mu) \to L_q(\nu)$ with $T 1_X = 1_Y$ for $1 < p < \infty$, $1 < q < \infty$. It is known that an operator $T$ in $K[L_p(\mu), L_q(\nu)]$ is an extreme point of this set if and only if $T$ is a lattice homomorphism [6, III.9.2]. Further characterizations of the extreme points of $K[L_\infty(\mu), L_\infty(\nu)]$ as operators which are multiplicative or which carry characteristic functions into characteristic functions are given by Phelps [5, 2.2]. We will characterize the extreme points of the set $K[L_p(\mu), L_q(\nu)]_G$ of all operators in $K[L_p(\mu), L_q(\nu)]$ which are invariant with respect to a semigroup $G$ of positive constant preserving contractions on $L_p(\mu)$ for $p < \infty$. These characterizations, including generalizations of the above mentioned results, are in part similar to those of the extreme points of $K[C(X), C(Y)]_G$ for compact spaces $X$ and $Y$, which have been stated by Converse, Namioka and Phelps [2, 5.3].

1. Preliminaries. Throughout suppose that $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{B}, \nu)$ are finite positive measure spaces. $1_A$ stands for the characteristic function of $A$. The convex set of all positive linear operators $T: L_p(\mu) \to L_q(\nu)$ with $T 1_X = 1_Y$ is denoted by $K[L_p(\mu), L_q(\nu)]$ for $1 < p < \infty$, $1 < q < \infty$. Let $G$ be a sub-semigroup of $K[L_p(\mu), L_p(\mu)]$. A linear operator $T: L_p(\mu) \to L_q(\nu)$ is called invariant if $TV = T$ for all $V \in G$. $K[L_p(\mu), L_q(\nu)]_G$ denotes the convex set of all invariant elements in $K[L_p(\mu), L_q(\nu)]$. Furthermore, we denote by $D$ the linear hull of the set $\{V f - f: f \in L_p(\mu), V \in G\}$ and by $F$ the fixed space of $G$, i.e. $F = \{f \in L_p(\mu): V f = f$ for all $V \in G\}$.

The key for the characterizations of extreme points is the following fact. If $G$ is a contractive semigroup, i.e. $\sup \{\|V\|: V \in G\} < 1$ and $p < \infty$, then the semigroup $\text{co}(G)^-$ has a zero element, where $\text{co}(G)^-$ denotes the closed convex hull of $G$ in the space of all continuous linear operators on $L_p(\mu)$,
endowed with the topology of simple convergence [4, 1.4 and 2.3]. The (unique) zero element \( P \) of \( \text{co}(G)^- \) is a positive contractive projection onto \( F \) with \( P 1_X = 1_X \) (cf. [6, III.7.2]). Furthermore, an operator \( T \in K[L_p(\mu), L_q(\nu)] \) is invariant if and only if \( TP = T \). This follows from the continuity of \( T \) (cf. [6, II.5.3]).

2. Extreme invariant operators. The following characterization, which holds without any further hypotheses on \( G \) and \( p \), is a special case of [3, Theorem 5].

**Theorem 1.** Suppose \( T \in K[L_p(\mu), L_q(\nu)]_G \). Then \( T \) is an extreme point of \( K[L_p(\mu), L_q(\nu)]_G \) if and only if \( \inf \{ T(|f - h|) : h \in R 1_X + D \} = 0 \) for each \( f \in L_p(\mu) \).

Before we can formulate the main result, we need the following information.

**Lemma.** If \( G \) is a contractive semigroup, \( p < \infty \), and \( P \) is the zero element of \( \text{co}(G)^- \), then \( F = L_p(\mathbb{1}^\perp) \) and \( P \) is the \( \mathfrak{A}_0 \)-conditional expectation with \( \mathfrak{A}_0 = \{ A \in \mathfrak{A} : 1_A \in F \} \).

**Proof.** Obviously \( F \) is a closed subspace of \( L_p(\mu) \) with \( 1_X \in F \). Furthermore, \( F \) is a sublattice. Let \( f \in F \) and \( K \in G \). Since \( f^+ > f \) and \( f^+ > 0 \) we have \( Vf^+ > Vf = f \) and \( Vf^+ > 0 \). Hence, \( Vf^+ > f^+ \) and this implies \( Vf^+ = f^+ \) because \( V \) is a contraction. The first assertion follows from the well-known characterization of closed sublattices of \( L_p(\mu) \) which contain \( 1_X \) (cf. [6, III.11.2]). In view of the above mentioned properties of \( P \), the second assertion is a result of Ando [1, (proof of) Theorem 2].

Let \( E_{\mathfrak{A}_0} \) denote the \( \mathfrak{A}_0 \)-conditional expectation operator.

**Theorem 2.** Suppose \( T \in K[L_p(\mu), L_q(\nu)]_G \) and \( p < \infty \). If \( G \) is a contractive semigroup, then the following assertions are equivalent:

(i) \( T \) is an extreme point of \( K[L_p(\mu), L_q(\nu)]_G \).

(ii) \( \inf \{ T(10 f) : t \in R \} = 0 \) for each \( f \in F \).

(iii) \( T(E_{\mathfrak{A}_0} f) = TfTh \) for each \( f \in L_p(\mu), h \in L_\infty(\mu) \).

(iv) \( T(fh) = TfTh \) for each \( f \in F, h \in L_\infty(\mu, \mathfrak{A}_0) \).

(v) \( T|F \) is a lattice homomorphism.

(vi) \( T \) carries \( \mathfrak{A}_0 \)-measurable characteristic functions into characteristic functions.

**Proof.** The equivalence of (i) and (ii) follows from [3, Theorem 6].

(i) \( \Rightarrow \) (iii). Clearly it is sufficient to prove that assertion (iii) holds for those \( h \in L_\infty(\mu) \) such that \( 0 < h < 1_X \). Assuming \( 0 < h < 1_X \), we define a map \( T_h : L_p(\mu) \to L_q(\nu) \) by \( T_h f = (E_{\mathfrak{A}_0} f) \cdot h - TfTh \). Then \( T_h \) is an invariant linear operator with \( T_h 1_X = 0 \). If \( f > 0 \), then \( (T + T_h)f = Tf(1_Y - Th) + T(E_{\mathfrak{A}_0} f) \cdot h > 0 \) and \( (T - T_h)f = T(E_{\mathfrak{A}_0} f) \cdot (1_X - h) + TfTh > 0 \). Thus \( T \pm T_h \in K[L_p(\mu), L_q(\nu)]_G \) holds. Since \( T \) is extreme, this implies \( T_h = 0 \).

(iii) \( \Rightarrow \) (iv) is obvious.
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(iv) $\Rightarrow$ (v). First let $f \in L_\infty(\mu|\mathcal{A}_0)$. The chain $(T|f|^2) = T(|f|^2) = T(f^2) = (Tf)^2 = |Tf|^2$ shows that $T|f| = T|f|$ holds. Since $L_\infty(\mu|\mathcal{A}_0)$ is dense in $F$ and furthermore, $T$ and the lattice operations on $L_p(\mu)$ and $L_q(\nu)$ are continuous, this implies that $T|F$ is a lattice homomorphism.

(v) $\Rightarrow$ (vi). For $A \in \mathcal{A}_0$ we obtain $T(1_A \wedge T1_{A^c}) = T(1_A \wedge 1_{A^c}) = T0 = 0$. Hence, $T1_A$ is a characteristic function.

(vi) $\Rightarrow$ (ii). Let $A \in \mathcal{A}_0$. Since $T(|1_A - t1_x|) = |1 - t|T1_A + |t|T1_{A^c}$, it follows that

$$\inf\{T(|1_A - t1_x|): t \in \mathbb{R}\} \leq T1_A \wedge T1_{A^c} = 0.$$ By virtue of the continuity of $T$ it is readily verified that $\inf\{T(|f - t1_x|): t \in \mathbb{R}\} = 0$ is valid for each $f \in F$.

**Corollary 1.** Suppose $T \in K[L_p(\mu), R]_G$. Under the above hypotheses on $G$ and $p$, $T$ is an extreme point of $K[L_p(\mu), R]_G$ if and only if $T1_A \in \{0, 1\}$ for each $A \in \mathcal{A}_0$.

In the following corollary an application to conditional expectations is given.

**Corollary 2.** Suppose that $\mathcal{A}_1$ is a $\sigma$-subalgebra of $\mathcal{A}_0$. Under the above hypotheses on $G$ and $p$, the operator $E_{\mathcal{A}_1}$ is an extreme point of $K[L_p(\mu), L_p(\mu)]_G$ if and only if for each $A \in \mathcal{A}_0$ there exists $B \in \mathcal{A}_1$ with $\mu(A \triangle B) = 0$.

**Proof.** Obviously $E_{\mathcal{A}_1} \in K[L_p(\mu), L_p(\mu)]_G$ holds. The "if" part. Let $A \in \mathcal{A}_0$. By assumption there exists $B \in \mathcal{A}_1$ with $\mu(A \triangle B) = 0$ such that $E_{\mathcal{A}_1}1_A = E_{\mathcal{A}_1}1_B = 1_B$. The assertion follows from Theorem 2.

The "only if" part. Let $A \in \mathcal{A}_0$. From Theorem 2 follows

$$\mu(A \cap C) = \int E_{\mathcal{A}_1}(1_A1_C)d\mu$$

$$= \int E_{\mathcal{A}_1}1_A E_{\mathcal{A}_1}1_C d\mu = \int 1_C E_{\mathcal{A}_1}1_A d\mu$$

for each $C \in \mathcal{A}_0$. Hence, $E_{\mathcal{A}_1}1_A = 1_A$. This yields the assertion.

**Remark.** Let $G \subset K[L_\infty(\mu), L_\infty(\mu)]$. Then $V \in G$ can be extended to a positive linear contraction on $L_1(\mu)$ if (and only if) $V$ is expectation invariant, i.e. $\int Vf d\mu = \int f d\mu$ for each $f \in L_\infty(\mu)$. Thus the preceding characterizations are valid for $p = \infty$ if $G$ is a semigroup of expectation invariant operators.

**References**


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