AN EXTREMAL PROBLEM FOR FUNCTIONS
OF POSITIVE REAL PART WITH APPLICATION
TO A RADIUS OF CONVEXITY PROBLEM

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Abstract. The functional \( \text{Re}\{zp'(z)/(p(z) + \beta + it)\} \) for \( \beta > -1 \), \( |z| < r \), \( 0 < r < 1 \), is minimized for all real \( t \) over the class of functions of positive real part. The result is applied to obtain the radius of convexity for a family of regular functions.

1. Introduction. Let \( S^*(\sigma), \sigma < 1 \), be the class of functions \( g(z) \) regular in \( A = \{z: |z| < 1\} \) such that \( g(0) = 0 \), \( g'(0) = 1 \) and \( \text{Re}\{zg'(z)/g(z)\} > \sigma \) for all \( z \in \Delta \).

We define the class \( K(\sigma, \lambda), \sigma < 1, \lambda < 1 \), to be the set of functions \( f(z) \) with \( f(0) = 0 \), \( f'(0) = 1 \), for which there exist \( g(z) \in S^*(\sigma) \) and a real number \( \alpha \), \( |\alpha| < \pi/2 \), such that

\[
\text{Re}\left\{ e^{i\alpha}\left[ zf'(z)/g(z) - \lambda \right] \right\} > 0 \quad \text{for} \quad z \in \Delta.
\]

The radius of convexity of \( K(\sigma, \lambda) \) is the greatest value of \( r, 0 < r < 1 \), for which \( \text{Re}\{1 + zf''(z)/f(z)\} > 0 \) for \( |z| < r \) and for all \( f(z) \) in \( K(\sigma, \lambda) \). Jablonski and Wesolowski [1] found a lower bound for the radius of convexity of \( K(0, \lambda) \) but the result is not sharp. Jankovics [2] obtained the radius of starlikeness of the class \( \{f(z): \text{Re}\{e^{i\alpha}\left[f(z)/z - \lambda\right]\} > 0, z \in \Delta\} \) by means of some results of Ruscheweyh [6]. The latter problem turns out to be equivalent to the former one with \( g(z) \equiv z \).

We approach the problem of finding the radius of convexity of \( K(\sigma, \lambda) \) by first determining

\[
M(\beta, r) = \min_{t \in \mathbb{R}} \left\{ M(\beta, r, t) \right\},
\]

where

\[
M(\beta, r, t) = \min_{p \in P} \left\{ \text{Re} \left[ \frac{zp'(z)}{p(z) + \beta + it} \right] \right\},
\]

for \( 0 < r < 1, \beta > 0 \) and \( P \) denoting the class of regular functions \( p(z) = 1 + \sum_{k=1}^\infty c_k z^k \) and \( \text{Re}\{p(z)\} > 0 \) for \( z \in \Delta \). This extremal problem is well known and interesting on its own. Robertson [5], by means of a variational

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method, showed that the extremal functions in (1) have the form

$$p_0(z) = \gamma \frac{1 + z e^{i \theta_1}}{1 - z e^{i \theta_1}} + (1 - \gamma) \frac{1 + z e^{i \theta_2}}{1 - z e^{i \theta_2}},$$

where $0 < \gamma < 1$, $0 < \theta_1 < 2\pi$, $0 < \theta_2 < 2\pi$ and the parameters $\gamma$, $\theta_1$, $\theta_2$ are to be determined. Zmorović [8] investigated $M(\beta, r, t)$ using Robertson's result but was only able to determine $M(0, r, t)$ and $M(\beta, r, 0)$ while Robertson [5] has previously found $M(0, r)$. We obtain $M(\beta, r)$ by minimizing the functional first with respect to $t$ and then over the class $P$.

2. The extremal problem. Before stating the main result of this section we need three lemmas. Using bilinear maps we may easily prove

**Lemma 1.** Let $a$ and $b$ be complex numbers such that $\text{Re } b > 0$ then

$$\text{Re} \left\{ \frac{a}{b + it} \right\} > \frac{\text{Re} \{a\} - |a|}{2\text{Re} \{b\}} \quad (t \in \mathbb{R}).$$

Next, let $H$ be the set of functions $f(z)$ regular in $\Delta$. A real functional $\phi$ defined on a convex subset $F$ of $H$ is said to be convex on $F$ if

$$\phi(\gamma x + (1 - \gamma)y) < \gamma \phi(x) + (1 - \gamma)\phi(y),$$

for any $\gamma$, $0 < \gamma < 1$, and $x, y$ in $F$. If equality always holds then $\phi$ is said to be affine on $F$.

A function $f_0 \in F$ is said to be an extreme point of $F$ if we cannot write $f_0 = \gamma f_1 + (1 - \gamma)f_2$, for some $\gamma$, $0 < \gamma < 1$, and distinct functions $f_1, f_2$ in $F$. Let $E_F$ denote the set of extreme points of $F$. Then we have

**Lemma 2.** [4]. Let $-\phi$ be a convex real functional on a compact convex subset $F$ of $H$. Then there always exists $f_1 \in E_F$ such that $\min_{f \in F} \phi(f) = \phi(f_1)$.

The next result is a generalization of a theorem by Ruscheweyh [7].

**Lemma 3.** Let $F$ be a compact convex set in $H$. Suppose that the real functionals $-\phi$ and $\psi$ are convex and affine on $F$ respectively. If $\psi(f) > 0$ for all $f \in F$, then $\min_{f \in F} \{\phi(f)/\psi(f)\} = \{\phi(f_1)/\psi(f_1)\}$, for some $f_1 \in E_F$.

**Proof.** Define $d = \min_{F} \{\phi(f)/\psi(f)\} = \phi(f_0)/\psi(f_0)$, some $f_0 \in F$. Clearly $0 < \phi(f) - d\psi(f)$ and thus $0 = \min_{F} \{\phi(f) - d\psi(f)\}$. But $d\psi(f) - \phi(f)$ is convex and hence by Lemma 2, there exists $f_1 \in E_F$ such that $0 = \phi(f_1) - d\psi(f_1)$, and so $d = \phi(f_1)/\psi(f_1)$.

**Theorem 1.** For $\beta > 0$, $p \in P$, $t \in \mathbb{R}$, $|z| < r$,

(a) $\text{Re} \left\{ \frac{zp(z)}{p(z) + \beta + it} \right\} > \frac{-2r}{(1 + r)[\beta(1 + r) + (1 - r)]}, \quad \text{for } r < r_0,$

(b) $\text{Re} \left\{ \frac{zp'(z)}{p(z) + \beta + it} \right\} > \frac{(1 + r)^2[(1 - r)^2 - u_1]}{2u_1(1 - r^2 + \beta u_1)}, \quad \text{for } r > r_0$,
where \( u_1 = (1 - r)^2(1 + \{1 + (1 + r)/\beta(1 - r)\}^{1/2}) \), and \( r_0 = (x_0 - 1)/(x_0 + 1) \), \( x_0 \) being the positive root of the equation \( \beta x^3 - 2\beta x - 1 = 0 \).

For \(-1 < \beta < 0, p \in P, t \in \mathbb{R}, |z| < r < (1 + \beta)/(1 - \beta)\), (a) also holds.

**Proof.** If \( \Re\{p(z) + \beta\} > 0 \) then from Lemma 1

\[
\begin{align*}
\min_{t \in \mathbb{R}} \Re \left( \frac{zp'(z)}{p(z) + \beta + it} \right) &= \frac{\Re\{zp'(z)\} - |zp'(z)|}{2 \Re\{p(z) + \beta\}}.
\end{align*}
\]

It can be verified that \(-\phi(p) \equiv |zp'(z)| - \Re\{zp'(z)\}\) is convex on \( P \) and that \( \psi(p) \equiv 2 \Re\{p(z) + \beta\}\) is affine on \( P \). Also, \( \psi(p) > 0 \) for all \( p \in P, |z| < r < 1 \) if \( \beta > 0 \), and \( \psi(p) > 0 \) for all \( p \in P, |z| < r < (1 + \beta)/(1 - \beta) \) if \(-1 < \beta < 0. \) Since \( P \) is compact and convex, we may apply Lemma 3 to obtain

\[
\begin{align*}
\min_{p \in P} \frac{\Re\{zp'(z)\} - |zp'(z)|}{2 \Re\{p(z) + \beta\}} &= \frac{\Re\{p_1(z) + \beta\}}{2 \Re\{p(z) + \beta\}},
\end{align*}
\]

for some \( p_1(z) \in E_p \). But

\[
E_p = \left\{ \frac{1 + ze^{i\theta}}{1 - ze^{i\theta}} : 0 < \theta < 2\pi \right\}.
\]

Thus,

\[
M(\beta, r) = \min_{0 < \theta < 2\pi} \frac{r(1 + r)^2(\cos \theta - 1)}{\beta(1 - 2r \cos \theta + r^2) + 1 - r^2}[1 - 2r \cos \theta + r^2].
\]

On making the substitution \( u = 1 - 2r \cos \theta + r^2 \), we can write

\[
M(\beta, r) = \min_{m < u < n} s(u),
\]

where \( m = (1 - r)^2, n = (1 + r)^2 \) and

\[
s(u) = n(m - u)/2u(1 - r^2 + \beta u).
\]

Now

\[
s'(u) = \frac{n}{2} \cdot \beta u^2 - 2m\beta u - m(1 - r^2) \cdot \frac{u^2(1 + \beta u - r^2)^2}{u^2(1 + \beta u - r^2)^2}.
\]

For \(-1 < \beta < 0 \) and \( r < (1 + \beta)/(1 - \beta) \), \( s'(u) < 0 \), therefore in this case we find

\[
M(\beta, r) = s(n) = -\frac{2r}{(1 + r)[\beta(1 + r) + (1 - r)]}.
\]

For \( \beta > 0 \), \( s'(u) \) has zeros at

\[
\frac{u_1}{u_2} = \frac{m \pm [m^2 + m(1 - r^2)/\beta]^{1/2}}{m^2 + m(1 - r^2)/\beta}.
\]

Since \( u_2 < 0 < m \) and \( u_1 > m \), \( M(\beta, r) = s(u_1) \) if \( u_1 < n \), otherwise \( M(\beta, r) \)
\[ s(n). \text{ The condition } u_1 > n \text{ is equivalent to} \]
\[ (1 - r)^2 + \left[ (1 - r)^4 + (1 - r)^2(1 - r^2)/\beta \right]^{1/2} > (1 + r)^2 \]
which, on putting \( x = (1 + r)/(1 - r) > 1 \), becomes
\[ 1 + (1 + x/\beta)^{1/2} > x^2. \]
Thus \( (1 + x/\beta)^{1/2} > x^2 - 1 > 0 \) and consequently \( u_1 > n \) if \( f(x) = \beta x^3 - 2\beta x - 1 > 0 \). Our result follows from the fact that \( f(x) \) has exactly one positive root, say \( x_0 \) in \([1, \infty)\).

3. Radius of convexity of \( K(\sigma, \lambda) \).

**Theorem 2.** The radius of convexity \( r_c \) of \( K(\sigma, \lambda) \) is given by
(a) the least positive root of
\[ 0 = \sigma + (1 - \sigma) \frac{1 - r}{1 + r} - \frac{2r}{(1 + r)[\beta(1 + r) + (1 - r)]} \]  \( \beta = \lambda/(1 - \lambda) \) (3)
when \( \lambda < \lambda_0 = (x^3 - 2x + 1)^{-1} \), where \( x = \sigma + (\sigma^2 - 2\sigma + 4)^{1/2} \),
(b) the least positive root of
\[ 0 = \sigma + (1 - \sigma) \frac{1 - r}{1 + r} + \frac{(1 + r)^2}{2} \frac{(1 - r^2 - u_1)}{u_1(1 - r^2 + \beta u_1)} \]  (4)
when \( \lambda > \lambda_0 \), where \( u_1 = (1 - r)^2[1 + (1 + (1 + r)/\beta(1 - r))^{1/2}] \). These results are sharp.

**Proof.** We have that \( f(z) \in K(\sigma, \lambda) \) if and only if \( |zf'(z)/g(z) - \lambda|/(1 - \lambda) \) is subordinate to
\[ 1 + cz \frac{1 + z}{1 - z} = \frac{1}{2} (1 - c) + \frac{1}{2} (1 + c) \frac{1 + z}{1 - z}, \]
where \( c = e^{-2\alpha} \). Hence, there exists \( p(z) \in P \) such that
\[ f'(z) = z^{-1}g(z)\left\{ \lambda + (1 - \lambda)[\frac{1}{2}(1 - c) + \frac{1}{2}(1 + c)p(z)] \right\}. \]
We thus find
\[ 1 + \frac{zf''(z)}{f'(z)} = \frac{zg'(z)}{g(z)} + \frac{zp'(z)}{(1 - c)/(1 + c)(1 - \lambda) + \lambda/(1 - \lambda) + p(z)}. \]
Now as \( c = e^{-2\alpha} \)
\[ \frac{\lambda}{1 - \lambda} + \frac{1 - c}{1 + c} \frac{1}{1 - \lambda} = \beta + it, \]
where \( \beta = \lambda/(1 - \lambda) \) and \( t = (\tan \alpha)/(1 - \lambda) \). It is clear that all possible values of \( t \) are taken as \( \alpha \) varies in \( (-\frac{1}{2}\pi, \frac{1}{2}\pi) \). Thus
\[ \text{Re} \left\{ 1 + \frac{zf''(z)}{f'(z)} \right\} = \text{Re} \left\{ \frac{zg'(z)}{g(z)} \right\} + \text{Re} \left\{ \frac{zp'(z)}{p(z) + \beta + it} \right\}. \]  (5)
For \( g(z) \in S^*(\sigma) \), it is well known that
\[
\Re \left\{ \frac{zg'(z)}{g(z)} \right\} \geq \sigma + (1 - \sigma) \frac{1 - r}{1 + r}.
\]
(6)

From (5), (6) and Theorem 1 we obtain the required result. It remains to determine \( \lambda_0 \). When \( \lambda = \lambda_0 \), we must have that the radii of convexity in the two cases coincide, say \( r_c = R \). Then, according to Theorem 1 and part (a) of this theorem, the values of \( R \) and \( \beta_0 = \lambda_0/(1 - \lambda_0) \) are determined by
\[
\beta_0 x^3 - 2\beta_0 x - 1 = 0, \quad x = (1 + R)/(1 - R) > 1,
\]
and
\[
0 = \sigma + (1 - \sigma) \frac{1 - R}{1 + R} - \frac{2R}{(1 + R)[\beta_0(1 + R) + (1 - R)]}.
\]
Elimination of \( \beta_0 \) and \( R \) from these equations gives
\[
(x^2 - 2\sigma x + 2\sigma - 4) = 0.
\]
It follows that \( x = \sigma + (\sigma^2 - 2\sigma + 4)^{1/2} \) and hence \( \lambda_0 \) is (uniquely) determined and the proof of Theorem 2 is completed.

**Remark 1.** For \( \lambda < \lambda_0 \), the extremal function is
\[
f_0(z) = \int_0^z \left[ \lambda + (1 - \lambda) \left( \frac{1 + \xi}{1 - \xi} \right)^2 \right] \frac{d\xi}{(1 - \xi)^{2(1 - \sigma)}}.
\]
For \( \lambda > \lambda_0 \), the extremal functions are
\[
f_0(z) = \int_0^z \left[ \lambda + \frac{1 - \lambda}{2} \left( 1 - c_0 \right) + \left( 1 + c_0 \right) \frac{1 + e^{i\theta \xi}}{1 - e^{i\theta \xi}} \right] \frac{d\xi}{(1 - \xi)^{2(1 - \sigma)}}
\]
where \( \cos \theta_0 = (1 + r^2 - u_1)/2r \) and \( c_0 = [(1 - \lambda)it_0 - 1]/[(1 - \lambda)it_0 + 1] \) with \( u_1 \) as defined in Theorem 1 and
\[
t_0 = \frac{2r \sin \theta_0}{u_1} \left\{ \frac{(1 - r^2 + \beta u_1)(1 - r)}{(1 + r)^2[(1 - r)^2 - u_1]} - 1 \right\}.
\]

Putting \( g(z) \equiv z \), we recover Jankovics’ result [2]. For \( \sigma = 0 \), our result differs from that of Jablonski and Wesolowski [1] as expected.

**Remark 2.** Libera [3] considered the class \( C(\sigma, \lambda) \) of functions \( f(z) \) such that \( \Re \{e^{inz}f(z)/g(z) \} > \lambda \) for some \( g(z) \in S^*(\sigma) \). Libera was only able to place a lower bound on the radius of convexity of \( C(\sigma, \lambda) \). Now, since \( C(\sigma, \lambda) \subseteq K(\sigma, \lambda) \), \( r_c \) for \( K(\sigma, \lambda) \) is another lower bound for the radius of convexity of \( C(\sigma, \lambda) \). However, for the case \( \lambda < \lambda_0 \), as the extremal function \( f_0(z) \) belongs to \( C(\sigma, \lambda) \), we conclude that \( r_c \) is also the radius of convexity for \( C(\sigma, \lambda) \).
REFERENCES


2. R. Jankovics, Über Funktionen mit der Eigenschaft \( \text{Re}(e^{i\theta}(f(z)/z - \beta)) > 0 \), Math. Z. 143 (1975), 235–242.


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