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CAPACITIES AND SPANS ON RIEMANN SURFACES[1]

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ABSTRACT. Let $K(z, z)$, $R(z, z)$, and $C(z)$ be the values of the Bergman kernel, the reduced Bergman kernel and the analytic capacity on an open Riemann surface Ω (with respect to a local parameter z). Let $M(z) = \pi K(z, z)$ and $S(z) = \sqrt{\pi R(z, z)}$. For $\Omega \notin O_G$ and for each integer $n > 0$, it is shown that

$$C^{(n+1)(n+2)} \leqslant (n+1)! \left(\prod_{k=0}^{n+1} k! \right)^{-2} \det\|M_{j\bar{k}}\|_{j,k=0}^{n},$$

where $C = C(z)$ and $M_{j\bar{k}} = (\partial^{j+k}/\partial z^j \partial \bar{z}^k) M(z)$. Equality occurs if and only if Ω is conformally equivalent to the unit disk less (possibly) a closed set of inner capacity zero. The special case of this result, namely when $n = 0$, is due to Hejhal and Suita. Let $\kappa(z)$ be the curvature of the "span metric" $S(z)|dz|$. As an attempt to resolve a conjecture of Suita, we also show that for $\Omega \notin O_{AD}$, $\kappa(z) \leqslant -2$ for each $z \in \Omega$. Both results are proved by studying suitable extremal problems.

1. Introduction. Let $K(z, z)$, $R(z, z)$, and $C(z)$ be the values of the *Bergman kernel*, the *reduced Bergman kernel*, and the *analytic capacity* on an open Riemann surface Ω (with respect to a local parameter z). We write $M(z) = \pi K(z, z)$ and $S(z) = \sqrt{\pi R(z, z)}$. Following Schiffer [4], $S(z)$ will be called the *span* of Ω at z. A problem originated by Sario and Oikawa [3, p. 342], and followed by others, notably by Hejhal [1, p. 106] and Suita [5], is to find relations amongst the quantities $M(z)$, $S(z)$, and $C(z)$. Concerning the relation between $M(z)$ and $C(z)$, Hejhal [1, p. 106] obtained an answer for finite Riemann surfaces Ω by showing that $M(z) > C^2(z)$ if Ω is not simply connected. The general case of this result was given by Suita [5]. He showed that for $\Omega \notin O_G$, $M(z) \geqslant C^2(z)$ with equality if and only if Ω is conformally equivalent to the unit disk less (possibly) a closed set of inner capacity zero.[2]

In the present paper we provide yet another generalization of Suita's result to include higher derivatives of $M(z)$ (compare [3, p. 114] and [2]). This result (Theorem 1), is proved by studying suitable extremal problems and employing in part a method of proof similar to that of Suita [5].

With regard to the span of Ω, Suita [5], led by an earlier special result of

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[2] This equality statement is different than that of [5] where it has been stated erroneously that equality can occur when $\Omega \in O_G$. However, if Ω is a parabolic Riemann surface of positive genus, there always exists a nontrivial analytic differential with a finite norm (cf. [3, pp. 246–249]). Hence in this case $M(z) > 0$ while $C(z) = 0$.

Zarankiewicz [6], has raised the following conjecture:

CONJECTURE 1. Let $\Omega \notin O_{AD}$ and let $\kappa(z)$ be the curvature of $S(z)|dz|$. Then $\kappa(z) \leqslant -4$ and equality holds if and only if Ω is conformally equivalent to the unit disk less (possibly) a closed set expressed as a countable union of N_D sets.

We were not able to resolve completely this conjecture. However, when Ω is an arbitrary plane region, we are able to show that $\kappa(z) \leqslant -2$ for all $z \in \Omega$ (Theorem 2). We also reformulate the above conjecture in, hopefully, a more manageable way (Conjecture 2).

2. A generalization. Let $\Omega \notin O_G$ and let $H_2(\Omega)$ be the space of square integrable analytic differentials on Ω. In what follows there is no essential loss of generality in assuming global coordinates z in Ω. Therefore, for sake of simplicity, we assume that $H_2(\Omega)$ is in fact the space of square integrable analytic functions f on Ω. This is a Hilbert space normed by

$$\|f\| = \left\{ \iint_\Omega |f(z)|^2 \, dx \, dy \right\}^{1/2}.$$

Since $\Omega \notin O_G$, $H_2(\Omega)$ possesses a nonconstant Bergmen kernel function $K(z, \zeta)$, $z, \zeta \in \Omega$.

Let U_z be an open set around a fixed point $z \in \Omega$. We write

$$A_m(z) = \left\{ f \in C^\infty(U_z) \colon (\partial^k/\partial z^k) f(z) = 0, k = 0, 1, \dots, m-1 \right\},$$

and

$$S_m(z) = \left\{ f \in H_2(\Omega) \colon \|f\|^2 \leqslant \pi \right\} \cap A_m(z), \qquad m \geqslant 1.$$

We also let

$$S_0(z) = \left\{ f \in H_2(\Omega) \colon \|f\|^2 \leqslant \pi \right\}$$

and note that $S_0(z)$ is in fact, independent of z. Clearly, $S_m(z)$, for any $m \geqslant 0$, is a nonempty closed convex subset of $H_2(\Omega)$.

For a fixed $\zeta \in \Omega$ and $m \geqslant 0$ we consider the following extremal problem

$$\lambda_m(\zeta) = \max\left\{ |f^{(m)}(\zeta)|^2 \colon f \in S_m(\zeta) \right\}. \tag{2.1}$$

This problem, of course, has (up to a rotation) a unique solution. Moreover, by a standard Hilbert space argument and a use of the reproducing property of $K(z, \zeta)$ one can show that

$$\lambda_m(\zeta) = J_m / J_{m-1}, \tag{2.2}$$

where $J_{-1} \equiv 1$ and

$$J_m = J_m(\zeta) = \det\|M_{j\bar{k}}\|_{j,k=0}^m, \qquad m \geqslant 0. \tag{2.3}$$

Here

$$M_{j\bar{k}} = \frac{\partial^{j+k}}{\partial \zeta^j \partial \bar{\zeta}^k} M(\zeta), \quad M_{0\bar{0}} = M(\zeta) = \pi K(\zeta, \zeta).$$

As customary $H_\infty(\Omega)$ stands for the Banach space of bounded holomorphic functions f on Ω normed by $\|f\|_\infty = \sup_{z\in\Omega}|f(z)|$. The analytic capacity is then given by

$$C = C(\zeta) = \max\{|f'(\zeta)|: f \in H_\infty(\Omega), \|f\|_\infty \leqslant 1, f(\zeta) = 0\}.$$

This maximum is uniquely attained (up to a rotation) by the Ahlfors function $F(z) = F(z: \zeta)$. Thus, $F(\zeta) = 0$ and $F'(\zeta) = C(\zeta)$.

Let

$$h_m(z) = [h(z)]^m, \quad h(z) = \exp[-(G(z, \zeta) + iG^*(z, \zeta))], \qquad m \geqslant 0,$$

be the multivalued function in Ω, where $G(z, \zeta)$ is the Green's function of Ω and $G^*(z, \zeta)$ is its harmonic conjugate. Then $|h_m(z)|$, $|h'_m(z)|$ and $h'_m(z)/h_m(z)$ are single valued, and, moreover

$$\|h'_m\|^2 = \iint_\Omega |h'_m(z)|^2\, dx\, dy = m\pi, \qquad m \geqslant 0.$$

We are now in a position to state our theorem. The special case when $n = 0$ is due to Hejhal [**1**, p. 106] and Suita [**5**].

THEOREM 1. *Let $\Omega \notin O_G$. For each integer $n \geqslant 0$ and $\zeta \in \Omega$ we have*

$$C^{(n+1)(n+2)} \leqslant (n+1)! \left(\prod_{k=0}^{n+1} k! \right)^{-2} \det\|M_{j\bar{k}}\|_{j,k=0}^{n}. \tag{2.4}$$

Equality holds if and only if Ω is conformally equivalent to the unit disk less (possibly) a closed set of inner capacity zero.

PROOF. Let $\Omega \notin O_G$ and consider the function

$$\phi_m(z) = \frac{1}{\sqrt{m+1}} [F(z)]^{m+1} \frac{h'_{m+1}(z)}{h_{m+1}(z)}, \qquad m \geqslant 0.$$

This function is analytic and single valued in Ω. Also, since $F^{m+1}/h_{m+1} = (F/h)^{m+1}$ is analytic in Ω, $|F^{m+1}/h_{m+1}| \leqslant 1$. Therefore,

$$\|\phi_m\|^2 = \left\| \frac{F^{m+1}}{h_{m+1}} \frac{h'_{m+1}}{\sqrt{m+1}} \right\|^2 \leqslant \left\| \frac{h'_{m+1}}{\sqrt{m+1}} \right\|^2 = \pi. \tag{2.5}$$

Further, by a direct computation, we obtain

$$\phi_m^{(k)}(\zeta) = \sqrt{m+1}\; m!\, C^{m+1} \delta_{km}, \qquad k = 0, 1, \ldots, m.$$

Consequently, $\phi_m \in S_m(\zeta)$ and hence, using (2.1),

$$|\phi_m^{(m)}(\zeta)|^2 = (m+1)(m!)^2 C^{2(m+1)} \leqslant \lambda_m(\zeta).$$

Therefore, according to (2.2),

$$(m+1)(m!)^2 C^{2(m+1)} \leqslant J_m / J_{m-1}.$$

Upon multiplying the above inequalities, running from $m = 0$ through $m = n$, and, using (2.3), we obtain (2.4). Equality in (2.4) entails equality in (2.5) for each $m = 0, 1, \ldots, n$. This implies that $|F^{m+1}/h_{m+1}| = |F/h|^{m+1} \equiv 1$ or that h is single valued. This holds, as in the proof of Suita [5], if and only if Ω is conformally equivalent to the unit disk less (possibly) a closed set of inner capacity zero. This concludes the proof of the theorem.

REMARKS. 1. It is easily verified that $C^{(n+1)(n+2)}/\det\|M_{j\bar{k}}\|_{j,k=0}^{n}$ is an absolute conformally invariant function.

2. It is clear that inequality (2.4) is also valid for $\Omega \in O_G$. In this case a trivial equality occurs when Ω is planar. If Ω is a parabolic Riemann surface of positive genus then for $n = 0$ we have a strict inequality namely, $M(z) > 0 = C(z)$. For $n \geqslant 1$, however, there exist cases for which trivial equality in (2.4) occurs.

3. The curvature of the span. Let $\Omega \notin O_{AD}$ be a plane region, and denote by $D(\Omega)$ the class of all holomorphic functions f in Ω for which the Dirichlet integral

$$D[f] = \iint_{\Omega} |f'(z)|^2 \, dx \, dy$$

does not exceed π. Then for $\zeta \in \Omega$ we have

$$S(\zeta) = \max\{|f'(\zeta)| : f \in \mathbf{D}(\Omega), f(\zeta) = 0\}.$$

The above maximum is uniquely attained (up to a rotation). We consider the conformally invariant metric $S(z)|dz|$. Its curvature is given by

$$\kappa(z) = -S^{-2}\Delta \log S, \quad S = S(z),$$

where Δ is the Laplacian operator. We note first the following straightforward lemma:

LEMMA 1. *For* $S = S(\zeta)$, $\zeta \in \Omega$, *we have*

$$2^{-1}S^2\Delta \log S = \max\{|f''(\zeta)|^2 : f \in \mathbf{D}(\Omega), f(\zeta) = f'(\zeta) = 0\},$$

and the maximum is uniquely attained (*up to a rotation*).

We first treat the finite case. For this purpose we let C_n designate the class of all regular regions which are bounded by n closed analytic disjoint curves. Thanks to the conformal invariance of the curvature we may assume, without loss of generality, that $\Omega \in C_n$. Moreover, for the same reason, we can also assume that $\zeta = \infty \in \Omega$ and that the area of the complement $E = \mathbf{C} - \Omega$, mE, is maximal amongst all regions Ω^* which are conformally equivalent to Ω. Under these assumptions the closed analytic curves of $\partial\Omega$ are convex and $mE = \pi S^2(\infty)$ (cf. [4]).

Let

$$p(z) = z + a/z + \cdots, \quad q(z) = z + b/z + \cdots,$$

be the horizontal and vertical slit functions, respectively, of Ω. According to Schiffer [4] $p(z) + q(z) = 2z$ and

$$p(z) - q(z) = \frac{2}{\pi} \iint_E \frac{du dv}{z - w}, \qquad z \in \Omega, w = u + iv. \tag{3.1}$$

It is well known that $a - b = 2S^2(\infty)$ and therefore,

$$mE = \pi S^2(\infty) = \pi(a - b)/2. \tag{3.2}$$

Also, $D[p - q] = 2\pi(a - b)$. We introduce the function

$$\phi(z) = \frac{p(z) - q(z)}{\sqrt{2(a - b)}}, \qquad z \in \Omega.$$

Clearly, $\phi(\infty) = 0$ and $D[\phi] = \pi$.

LEMMA 2. $|\phi(z)| \leqslant 1$ *for all* $z \in \Omega$.

PROOF. Using a method of Ahlfors-Beurling (see [3, p. 177]) we can show that

$$\left| \iint_E \frac{du dv}{z - w} \right| \leqslant \sqrt{\pi m E}$$

and the lemma then follows from (3.1) and (3.2).

We now define a second function

$$\psi(z) = \tfrac{1}{2}[\phi(z)]^2, \qquad z \in \Omega.$$

ψ maps Ω into a subset of $\{z \colon |z| < 1\}$ and the convex closed analytic curves of $\partial\Omega$ are mapped onto curves of the same nature. For $n > 1$, ψ is not univalent but at most $2n$-valent with $\psi(\infty) = \psi'(\infty) = 0$. Also, ϕ is at most n-valent [3, p. 142]. It is, therefore, very plausible (as indeed is the case when $n = 1$) that the total image area of Ω under ϕ^2 is at most twice of that under ϕ. That is, $D[\phi^2] \leqslant 2D[\phi] = 2\pi$ or $D[\psi] \leqslant \pi/2$. If this were true then it would be possible to resolve Conjecture 1 as the proof of Theorem 2 will show. We are led, therefore, to the following reformulation of Conjecture 1:

CONJECTURE 2. Let $\Omega \in C_n$ be a maximal region as before. Then $D[\psi] \leqslant \pi/2$ with equality holding if and only if $n = 1$.

The case $n = 2$ of the above conjecture is readily verified by the earlier work of Zarankiewicz [6]. Here, however, we prove the following:

THEOREM 2. *Let* $\Omega \notin O_{AD}$ *be a plane region. Then* $\kappa(\zeta) \leqslant -2$ *for each* $\zeta \in \Omega$.

PROOF. We first show that $\kappa(\zeta) \leqslant -2$ when $\Omega \in C_n$ for any integer $n \geqslant 1$. Again, we may assume that Ω is maximal and $\zeta = \infty \in \Omega$. Let ϕ and ψ be as before. According to Lemma 2,

$$D[\psi] = \tfrac{1}{4} D[\phi^2] = \iint_\Omega |\phi(z)|^2 |\phi'(z)|^2 \, dx \, dy$$

$$\leqslant \iint_\Omega |\phi'(z)|^2 \, dx \, dy = D[\phi] = \pi.$$

Therefore, $\psi \in \mathbf{D}(\Omega)$, and, since $\psi(\infty) = \psi'(\infty) = 0$, Lemma 1 shows that

$$2^{-1}S^2\Delta \log S \geqslant |\psi''(\infty)|^2, \qquad S = S(\infty).$$

But, using (3.2),

$$\psi''(\infty) = [\phi'(\infty)]^2 = \frac{a-b}{2} = S^2$$

and so $2 \leqslant S^{-2}\Delta \log S$ or $\kappa(\infty) \leqslant -2$. This proves the theorem when $\Omega \in C_n$, $n \geqslant 1$ is any integer. For the general case $\Omega \notin O_{AD}$ we let $\{\Omega_m\}$ be a canonical exhaustion of Ω such that $\partial\Omega_m$ consists of a finite number of analytic curves. Then $\kappa(\zeta) = \lim_{m\to\infty} \kappa_{\Omega_m}(\zeta)$ and since $\kappa_{\Omega_m}(\zeta) \leqslant -2$ for each m the theorem follows.

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