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POSITIVELY CURVED COMPLEX SUBMANIFOLDS IMMERSED IN A COMPLEX SPACE FORM

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Dedicated to Professor T. Otsuki on his 60th birthday

ABSTRACT. The author gives the partial solution for the conjecture; a Kaehler submanifold in a complex space form of constant holomorphic sectional curvature 1 is totally geodesic if ever its holomorphic sectional curvature is greater than $\frac{1}{2}$.

Let $\tilde{M}_m(\tilde{c})$ be an m-dimensional complex space form of constant holomorphic sectional curvature $\tilde{c}$. Let M_n be an n-dimensional Kaehler submanifold of $\tilde{M}_m(\tilde{c})$. Then we have had the following conjecture:

If all holomorphic sectional curvatures of M_n are greater than $\tilde{c}/2$, then M_n is totally geodesic.

In [3], K. Ogiue gave some partial results for this conjecture.

In this paper, we shall prove the following partial solution for the above conjecture,

THEOREM. *Let M be an n-dimensional complete Kaehler submanifold immersed in an m-dimensional complex space form of constant holomorphic sectional curvature $\tilde{c}$. If every holomorphic sectional curvature of M (with respect to the induced metric) is greater than $3n\tilde{c}/(4n+2)$, then M is totally geodesic.*

1. Preliminaries. Let $\tilde{M}$ be an $(n+p)$-dimensional complex space form of constant holomorphic sectional curvature $\tilde{c}$ and M be an n-dimensional Kaehler submanifold immersed in $\tilde{M}$ (i.e., complex submanifold with the induced Kaehler structure). Let J (resp. $\tilde{J}$) be the complex structure of M (resp. $\tilde{M}$) and g (resp. $\tilde{g}$) be the Kaehler metric of M (resp. $\tilde{M}$). We denote by ∇ (resp. $\tilde{\nabla}$) the covariant differentiation with respect to g (resp. $\tilde{g}$). Then the second fundamental form σ of the immersion is given by

$$\sigma(X, Y) = \tilde{\nabla}_X Y - \nabla_X Y \quad \text{for vector fields } X \text{ and } Y \text{ on } M, \tag{1.1}$$

and it satisfies

$$\sigma(X, Y) = \sigma(Y, X), \quad \sigma(JX, Y) = \sigma(X, JY) = \tilde{J}(\sigma(X, Y)). \tag{1.2}$$

We choose a local field of orthonormal frames $e_1, e_2, \ldots, e_n, e_{1*} = Je_1, \ldots, e_{n*} = Je_n, e_{\bar{1}}, e_{\bar{2}}, \ldots, e_{\bar{p}}, e_{\bar{1}*} = \tilde{J}e_{\bar{1}}, \ldots, e_{\bar{p}*} = \tilde{J}e_{\bar{p}}$ in $\tilde{M}$ in such a way that, restricted to M, $e_1, \ldots, e_{n*}$ are tangent to M. With respect to the

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frame fields chosen above, let $\omega_1, \ldots, \omega_n, \omega_{1^*}, \ldots, \omega_{n^*}, \omega_{\tilde{1}}, \ldots, \omega_{\tilde{p}}, \omega_{\tilde{1}^*}, \ldots, \omega_{\tilde{p}^*}$ be the field of dual frames. Then, restricting these forms to M, we have the structure equations of M:[1]

$$\begin{cases} \omega_\lambda = 0, \quad \omega_{i\lambda} = \sum h_{ij}^\lambda \omega_j, \quad h_{ij}^\lambda = h_{ji}^\lambda, \\ d\omega_i = \sum \omega_{ij} \wedge \omega_j, \quad \omega_{ij} + \omega_{ji} = 0, \\ d\omega_{ij} = \sum \omega_{ik} \wedge \omega_{kj} - \Omega_{ij}, \quad \Omega_{ij} = \frac{1}{2} \sum R_{ijkl} \omega_k \wedge \omega_l. \end{cases} \tag{1.3}$$

The equation of Gauss is written as

$$\begin{aligned} R_{ijkl} = \frac{\tilde{c}}{4} &(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk} + J_{ik}J_{jl} - J_{il}J_{jk} + 2J_{ij}J_{kl}) \\ &+ \sum (h_{ik}^\lambda h_{jl}^\lambda - h_{il}^\lambda h_{jk}^\lambda), \end{aligned} \tag{1.4}$$

where

$$J = (J_{ij}) = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix},$$

I_n being the identity matrix. Let $H_\alpha = (h_{ij}^\alpha)$, $H'_\alpha = (h_{ab}^\alpha)$ and $H''_\alpha = (h_{ab^*}^\alpha)$, then we can easily see that

$$H_\alpha = \begin{pmatrix} H'_\alpha & H''_\alpha \\ H''_\alpha & -H'_\alpha \end{pmatrix} \quad \text{and} \quad H_{\alpha^*} = \begin{pmatrix} -H''_\alpha & H'_\alpha \\ H'_\alpha & H''_\alpha \end{pmatrix}. \tag{1.5}$$

Now, we can consider the following nonnegative functions on M;

$$K_N = \sum \left(\sum_k (h_{ik}^\lambda h_{kj}^\mu - h_{jk}^\lambda h_{ki}^\mu) \right)^2 \quad \text{and} \quad L_N = \sum h_{ij}^\lambda h_{ij}^\mu h_{kl}^\lambda h_{kl}^\mu.$$

Then we know the following differential equation [1]:

$$\frac{1}{2} \Delta \|\sigma\|^2 = \sum \|\nabla' \sigma\|^2 - K_N - L_N + \frac{n+2}{2} \tilde{c} \|\sigma\|^2, \tag{1.6}$$

where $\|\phi\|$ is the length of a form ϕ and ∇' is the covariant differentiation respect to the connection in (tangent bundle of M) $\oplus$ (normal bundle) induced naturally from $\tilde{\nabla}$.

2. Proof of Theorem. First of all we note that, by the theorem of Myers [3], M is compact. Let c be the minimum of all holomorphic sectional curvatures of M. The holomorphic sectional curvature $H(X)$ of M determined by a unit vector X is given by

$$H(X) = \tilde{c} - 2\|\sigma(X, X)\|^2. \tag{2.1}$$

[1]We use the following convention on the range of indices unless otherwise stated: $a, b, c = 1, 2, \ldots, n$; $i, j, k, l = 1, 2, \ldots, n, 1^*, \ldots, n^*$; $\alpha, \beta, \gamma = \tilde{1}, \ldots, \tilde{p}$; $\lambda, \mu, \nu = 1, 2, \ldots, \tilde{p}, 1^*, 2^*, \ldots, \tilde{p}^*$; and we agree that repeated indices under a summation sign without indication are summed over the respective range.

It is clear from (2.1) that $\tilde{c} \geqslant c$. Setting $A_{ij} := \sigma(e_i, e_j)$, from (1.2) we have

$$A_{a^*b^*} = -A_{ab}, \quad A_{ab^*} = A_{a^*b} = \tilde{J}A_{ab}, \quad \text{for } a, b. \tag{2.2}$$

For a vector $X = \lambda e_i + \eta e_j$ on M, we get

$$\sigma(X, X) = \lambda^2 A_{ii} + 2\lambda\eta A_{ij} + \eta^2 A_{jj}.$$

Since $H(X) \geqslant c$ for all unit vector X, $\|\sigma(X, X)\| \leqslant (\tilde{c} - c)/2$, in particular,

$$\|A_{aa}\|^2 = \|\sigma(e_a, e_a)\|^2 \leqslant (\tilde{c} - c)/2 \quad \text{for } \forall a. \tag{2.3}$$

Since $\|\sigma(X, X)\| \leqslant (\tilde{c} - c)/2$ for all unit vectors $X = \lambda e_i + \eta e_j$ $(i \neq j)$, given by

i	j	λ	η
a	b	λ	η
		λ	$-\eta$
a	b^*	λ	η
		λ	$-\eta$

i	j	λ	η
b	a	λ	η
		λ	$-\eta$
b	a^*	λ	η
		λ	$-\eta$

we have

$$(\lambda^4 + \eta^4)\left(\|A_{aa}\|^2 + \|A_{bb}\|^2\right) + 8\lambda^2\eta^2\|A_{ab}\|^2 \leqslant \tilde{c} - c. \tag{2.4}$$

Since $\lambda^4 + \eta^4 \geqslant 2\lambda^2\eta^2$, from (2.4) we get

$$\lambda^2\eta^2\left(\|A_{aa}\|^2 + \|A_{bb}\|^2 + 4\|A_{ab}\|^2\right) \leqslant (\tilde{c} - c)/2$$

in which, setting $\lambda^2 = \eta^2 = \frac{1}{2}$, we get

$$\|A_{aa}\|^2 + \|A_{bb}\|^2 + 4\|A_{ab}\|^2 \leqslant 2(\tilde{c} - c). \tag{2.5}$$

Since $\|A_{aa}\|^2 + \|A_{bb}^2\| \geqslant 2\|A_{aa}\| \cdot \|A_{bb}\|$, (2.5) implies

$$\|A_{aa}\| \cdot \|A_{bb}\| + 2\|A_{ab}\|^2 \leqslant \tilde{c} - c. \tag{2.6}$$

Using (2.3), (2.6) and the fact that $|\langle A_{aa}, A_{bb}\rangle| \leqslant \|A_{aa}\| \cdot \|A_{bb}\|$, we can prove

LEMMA. *The following inequality holds on M;*

$$K_N + L_N \leqslant (2n + 1)(\tilde{c} - c)\|\sigma\|^2. \tag{2.7}$$

PROOF OF LEMMA. For each α, we set

$$K_N^\alpha := \sum_{i,j,\mu}\left(\sum_k (h_{ik}^\alpha h_{kj}^\mu - h_{jk}^\alpha h_{ki}^\mu)\right)^2 + \sum_{i,j,\mu}\left(\sum_k \left(h_{ik}^{\alpha^*} h_{kj}^\mu - h_{jk}^{\alpha^*} h_{ki}^\mu\right)\right)^2,$$

$$L_N^\alpha := \sum_{i,j,k,l,\mu} h_{ij}^\alpha h_{ij}^\mu h_{kl}^\alpha h_{kl}^\mu + \sum_{i,j,k,l,\mu} h_{ij}^{\alpha^*} h_{ij}^\mu h_{kl}^{\alpha^*} h_{kl}^\mu.$$

Then $K_N = \sum_\alpha K_N^\alpha$ and $L_N = \sum_\alpha L_N^\alpha$. For fixed α, choosing a frame $\{e_1, \ldots, e_n, e_{1^*}, \ldots, e_{n^*}\}$ such that $h_{ij}^\alpha = \lambda_i^\alpha \delta_{ij}$, from (1.5) we have

$$H_\alpha = \begin{pmatrix} H'_\alpha & 0 \\ 0 & -H'_\alpha \end{pmatrix}, \quad H_{\alpha*} = \begin{pmatrix} 0 & H'_\alpha \\ H'_\alpha & 0 \end{pmatrix} \quad \text{and} \quad H'_\alpha = \begin{bmatrix} \lambda_1^\alpha & & 0 \\ & \ddots & \\ 0 & & \lambda_n^\alpha \end{bmatrix}. \tag{2.8}$$

Using (2.8), we have

$$K_N^\alpha = 8 \sum_{a \neq b} \left\{ (\lambda_a^\alpha)^2 + (\lambda_b^\alpha)^2 \right\} \|A_{ab}\|^2 + 16 \sum_a (\lambda_a^\alpha)^2 \|A_{aa}\|^2,$$

$$L_N^\alpha = 8 \sum_{a \neq b} \lambda_a^\alpha \lambda_b^\alpha \langle A_{aa}, A_{bb} \rangle + 8 \sum_a (\lambda_a^\alpha)^2 \|A_{aa}\|^2,$$

and so

$$K_N^\alpha + L_N^\alpha = 8 \sum_{a \neq b} \left[\left\{ (\lambda_a^\alpha)^2 + (\lambda_b^\alpha)^2 \right\} \|A_{ab}\|^2 + \lambda_a^\alpha \lambda_b^\alpha \langle A_{aa}, A_{bb} \rangle \right] + 24 \sum_a (\lambda_a^\alpha)^2 \|A_{aa}\|^2. \tag{2.9}$$

Since we have the following inequalities

$$2\lambda_a^\alpha \lambda_b^\alpha \langle A_{aa}, A_{bb} \rangle \leqslant -\left\{ (\lambda_a^\alpha)^2 + (\lambda_b^\alpha)^2 \right\} \langle A_{aa}, A_{bb} \rangle \quad \text{if } \langle A_{aa}, A_{bb} \rangle < 0,$$

$$2\lambda_a^\alpha \lambda_b^\alpha \langle A_{aa}, A_{bb} \rangle \leqslant \left\{ (\lambda_a^\alpha)^2 + (\lambda_b^\alpha)^2 \right\} \langle A_{aa}, A_{bb} \rangle \quad \text{if } \langle A_{aa}, A_{bb} \rangle \geqslant 0,$$

and $|\langle A_{aa}, A_{bb} \rangle| \leqslant \|A_{aa}\| \cdot \|A_{bb}\|$, we get

$$2\lambda_a^\alpha \lambda_b^\alpha \langle A_{aa}, A_{bb} \rangle \leqslant \left\{ (\lambda_a^\alpha)^2 + (\lambda_b^\alpha)^2 \right\} \|A_{aa}\| \cdot \|A_{bb}\|. \tag{2.10}$$

It follows from (2.9) and (2.10) that we have

$$K_N^\alpha + L_N^\alpha \leqslant 4 \sum_{a \neq b} \left\{ (\lambda_a^\alpha)^2 + (\lambda_b^\alpha)^2 \right\} \left\{ 2\|A_{ab}\|^2 + \|A_{aa}\| \cdot \|A_{bb}\| \right\} + 24 \sum_a (\lambda_a^\alpha)^2 \|A_{aa}\|^2. \tag{2.11}$$

Hence, using (2.3) and (2.6), from (2.11) we obtain

$$\begin{aligned} K_N^\alpha + L_N^\alpha &\leqslant 4(\tilde{c} - c) \left[\sum_{a \neq b} \left\{ (\lambda_a^\alpha)^2 + (\lambda_b^\alpha)^2 \right\} + 3 \sum_a (\lambda_a^\alpha)^2 \right] \\ &= 4(2n+1)(\tilde{c} - c) \sum_a (\lambda_a^\alpha)^2 \\ &= (2n+1)(\tilde{c} - c)(\operatorname{Tr} H_\alpha^2 + \operatorname{Tr} H_{\alpha*}^2). \end{aligned} \tag{2.12}$$

Since $K_N + L_N = \Sigma_\alpha (K_N^\alpha + L_N^\alpha)$ and $\|\sigma\|^2 = \Sigma_\alpha (\operatorname{Tr} H_\alpha^2 + \operatorname{Tr} H_{\alpha*}^2)$, (2.12) implies (2.7). Q.E.D.

PROOF OF THEOREM. Using (2.7), from (1.6) we have

$$\begin{aligned} \frac{1}{2} \Delta \|\sigma\|^2 &\geqslant \|\nabla' \sigma\|^2 + \frac{n+2}{2} \tilde{c} \|\sigma\|^2 - (2n+1)(\tilde{c} - c)\|\sigma\|^2 \\ &\geqslant (2n+1)\left(c - \frac{3n\tilde{c}}{4n+2} \right) \|\sigma\|^2 \quad \text{on } M. \end{aligned} \tag{2.13}$$

Since M is compact and $c > 3n\tilde{c}/(4n+2)$, from (2.13) $\Delta\|\sigma\|^2 = 0$ on M. Hence we have $\|\sigma\| = 0$ identically on M, so that M is totally geodesic. Q.E.D.

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