POSITIVELY CURVED COMPLEX SUBMANIFOLDS
IMMERSED IN A COMPLEX SPACE FORM

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Dedicated to Professor T. Otsuki on his 60th birthday

Abstract. The author gives the partial solution for the conjecture; a
Kaehler submanifold in a complex space form of constant holomorphic
sectional curvature 1 is totally geodesic if ever its holomorphic sectional
curvature is greater than $\frac{1}{3}$.

Let $\tilde{M}_m(\tilde{c})$ be an $m$-dimensional complex space form of constant
holomorphic sectional curvature $\tilde{c}$. Let $M_n$ be an $n$-dimensional Kaehler
submanifold of $\tilde{M}_m(\tilde{c})$. Then we have had the following conjecture:

If all holomorphic sectional curvatures of $M_n$ are greater than $\tilde{c}/2$, then $M_n$ is
totally geodesic.

In [3], K. Ogiue gave some partial results for this conjecture.

In this paper, we shall prove the following partial solution for the above
conjecture,

Theorem. Let $M$ be an $n$-dimensional complete Kaehler submanifold immer-
sed in an $m$-dimensional complex space form of constant holomorphic sectional
curvature $\tilde{c}$. If every holomorphic sectional curvature of $M$ (with respect to the
induced metric) is greater than $3\tilde{c}/(4n + 2)$, then $M$ is totally geodesic.

1. Preliminaries. Let $\tilde{M}$ be an $(n + p)$-dimensional complex space form of
constant holomorphic sectional curvature $\tilde{c}$ and $M$ be an $n$-dimensional
Kaehler submanifold immersed in $\tilde{M}$ (i.e., complex submanifold with the
induced Kaehler structure). Let $J$ (resp. $\tilde{J}$) be the complex structure of $M$
(resp. $\tilde{M}$) and $g$ (resp. $\tilde{g}$) be the Kaehler metric of $M$ (resp. $\tilde{M}$). We denote by
$\nabla$ (resp. $\tilde{\nabla}$) the covariant differentiation with respect to $g$ (resp. $\tilde{g}$). Then the
second fundamental form $\sigma$ of the immersion is given by

$$\sigma(X, Y) = \tilde{\nabla}_X Y - \nabla_X Y \quad \text{for vector fields } X \text{ and } Y \text{ on } M,$$

and it satisfies

$$\sigma(X, Y) = \sigma(Y, X), \quad \sigma(JX, Y) = \sigma(X, JY) = \tilde{J} (\sigma(X, Y)).$$

We choose a local field of orthonormal frames $e_1, e_2, \ldots, e_n, e_1^* =
Je_1, \ldots, e_n^* = Je_n, e_1, e_2, \ldots, e_p, e_1^* = \tilde{J}e_1, \ldots, e_p^* = \tilde{J}e_p$ in $\tilde{M}$ in such a
way that, restricted to $M$, $e_1, \ldots, e_n$ are tangent to $M$. With respect to the

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frame fields chosen above, let \( \omega_1, \ldots, \omega_n, \omega_1^*, \ldots, \omega_n^*, \omega_1, \ldots, \omega_n, \omega_1^*, \ldots, \omega_n^* \) be the field of dual frames. Then, restricting these forms to \( M \), we have the structure equations of \( M \):\(^1\)

\[
\begin{align*}
\omega_\lambda &= 0, \quad \omega_\lambda = \sum h^\lambda_j \omega_j, \quad h^\lambda_j = h^\lambda_j, \\
d\omega_i &= \sum \omega_{ij} \wedge \omega_j, \quad \omega_i + \omega_{ji} = 0, \\
d\omega_{ij} &= \sum \omega_{ik} \wedge \omega_{kj} - \Omega_{ij}, \quad \Omega_{ij} = \frac{1}{2} \sum R_{ijkl} \omega_k \wedge \omega_l.
\end{align*}
\]

The equation of Gauss is written as

\[
R_{ijkl} = \frac{\bar{c}}{4} \left( \delta_{ik} \delta_{jl} - \delta_{il} \delta_{jk} + J_{ik} J_{jl} - J_{il} J_{jk} + 2 J_{ij} J_{kl} \right) + \sum \left( h^\lambda_{ik} h_j^\lambda - h^\lambda_j h_{ik} \right),
\]

where

\[
J = (J_{ij}) = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix},
\]

\( I_n \) being the identity matrix. Let \( H_a = (h^a_{ij}), H'_a = (h^a_{ab}) \) and \( H''_a = (h''_{ab}) \), then we can easily see that

\[
H_a = \begin{pmatrix} H'_a & H''_a \\ H''_a & -H'_a \end{pmatrix} \quad \text{and} \quad H^* = \begin{pmatrix} -H''_a & H'_a \\ H'_a & H''_a \end{pmatrix}.
\]

Now, we can consider the following nonnegative functions on \( M \);

\[
K_N = \sum \left( \sum_k \left( h^\lambda_{ik} h_j^\lambda - h^\lambda_j h_{ik} \right) \right)^2 \quad \text{and} \quad L_N = \sum h^\lambda_{ik} h_j^\lambda h_{ik} h_{mj}.
\]

Then we know the following differential equation \([1]\):

\[
\frac{1}{2} \Delta \|\sigma\|^2 = \| \nabla' \sigma \|^2 - K_N - L_N + \frac{n+2}{2} \bar{c} \|\sigma\|^2,
\]

where \( \|\phi\| \) is the length of a form \( \phi \) and \( \nabla' \) is the covariant differentiation respect to the connection in (tangent bundle of \( M \)) \( \oplus \) (normal bundle) induced naturally from \( \tilde{\nabla} \).

2. Proof of Theorem. First of all we note that, by the theorem of Myers \([3]\), \( M \) is compact. Let \( c \) be the minimum of all holomorphic sectional curvatures of \( M \). The holomorphic sectional curvature \( H(X) \) of \( M \) determined by a unit vector \( X \) is given by

\[
H(X) = \bar{c} - 2 \|\sigma(X, X)\|^2.
\]

\(^1\)We use the following convention on the range of indices unless otherwise stated: \( a, b, c = 1, 2, \ldots, n; \quad i, j, k, l = 1, 2, \ldots, n; \quad \alpha, \beta, \gamma = 1, \ldots, \tilde{p}; \quad \lambda, \mu, \nu = 1, 2, \ldots, \tilde{p}, 1^*, 2^*, \ldots, \tilde{p}^*; \) and we agree that repeated indices under a summation sign without indication are summed over the respective range.
It is clear from (2.1) that \( \bar{c} > c \). Setting \( A_{ij} := (e_i, e_j) \), from (1.2) we have
\[
A_{ij} = - A_{ji}, \quad A_{ab} = A_{a'b'} = \bar{J} A_{ab}, \quad \text{for } a, b.
\] (2.2)

For a vector \( X = \lambda e_i + \eta e_j \) on \( M \), we get
\[
\sigma(X, X) = \lambda^2 A_{ii} + 2\lambda \eta A_{ij} + \eta^2 A_{jj}.
\]

Since \( H(X) > c \) for all unit vector \( X \), \( \|\sigma(X, X)\| < (\bar{c} - c)/2 \), in particular,
\[
\|A_{aa}\|^2 = \|\sigma(e_a, e_a)\|^2 < (\bar{c} - c)/2 \quad \text{for } \forall a.
\] (2.3)

Since \( \|\sigma(X, X)\| < (\bar{c} - c)/2 \) for all unit vectors \( X = \lambda e_i + \eta e_j \) \((i \neq j)\), given by

\[
\begin{array}{c|c|c|c|c|}
  i & j & \lambda & \eta \\
  \hline
  a & b & \lambda & \eta \\
  & & \lambda & -\eta \\
  a & b^* & \lambda & \eta \\
  & & \lambda & -\eta \\
  b & a & \lambda & \eta \\
  & & \lambda & -\eta \\
\end{array}
\]

we have
\[
(\lambda^4 + \eta^4)\left(\|A_{aa}\|^2 + \|A_{bb}\|^2\right) + 8\lambda^2 \eta^2 \|A_{ab}\|^2 \leq \bar{c} - c.
\] (2.4)

Since \( \lambda^4 + \eta^4 > 2\lambda^2 \eta^2 \), from (2.4) we get
\[
\lambda^2 \eta^2 \left(\|A_{aa}\|^2 + \|A_{bb}\|^2 + 4\|A_{ab}\|^2\right) < (\bar{c} - c)/2
\]
in which, setting \( \lambda^2 = \eta^2 = \frac{1}{2} \), we get
\[
\|A_{aa}\|^2 + \|A_{bb}\|^2 + 4\|A_{ab}\|^2 < 2(\bar{c} - c).
\] (2.5)

Since \( \|A_{aa}\|^2 + \|A_{bb}\|^2 > 2\|A_{aa}\| \cdot \|A_{bb}\| \), (2.5) implies
\[
\|A_{aa}\| \cdot \|A_{bb}\| + 2\|A_{ab}\|^2 < \bar{c} - c.
\] (2.6)

Using (2.3), (2.6) and the fact that \( \langle A_{aa}, A_{bb} \rangle < \|A_{aa}\| \cdot \|A_{bb}\| \), we can prove

**LEMMA.** The following inequality holds on \( M \);
\[
K_N + L_N < (2n + 1)(\bar{c} - c)\|a\|^2.
\] (2.7)

**PROOF OF LEMMA.** For each \( \alpha \), we set
\[
K_N^\alpha := \sum_{i,j,k} \left( \sum_{h,k} (h_{ik}^\alpha h_{kj}^\alpha - h_{ik}^\alpha h_{kj}^\alpha) \right)^2 + \sum_{i,j,k} \left( \sum_{h,k} (h_{ik}^\alpha h_{kj}^\alpha - h_{ik}^\alpha h_{kj}^\alpha) \right)^2,
\]
\[
L_N^\alpha := \sum_{i,j,k,l} h_{ij}^\alpha h_{kl}^\alpha h_{ik}^\alpha h_{lj}^\alpha + \sum_{i,j,k,l} h_{ij}^\alpha h_{kl}^\alpha h_{ik}^\alpha h_{lj}^\alpha.
\]

Then \( K_N = \sum_\alpha K_N^\alpha \) and \( L_N = \sum_\alpha L_N^\alpha \). For fixed \( \alpha \), choosing a frame \( \{e_1, \ldots, e_n, e_1^*, \ldots, e_n^*\} \) such that \( h_{ij}^\alpha = \lambda^{\alpha} \delta_{ij} \), from (1.5) we have

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$H_a = \begin{pmatrix} H_a' & 0 \\ 0 & -H_a' \end{pmatrix}, \quad H_{a^*} = \begin{pmatrix} 0 & H_a' \\ H_a' & 0 \end{pmatrix}$ and $H_a' = \begin{pmatrix} \lambda_a^* & 0 \\ 0 & \lambda_{a^*} \end{pmatrix}.$

(2.8)

Using (2.8), we have

$$K_N^a = 8 \sum_{a \neq b} \{(\lambda_a^a)^2 + (\lambda_b^b)^2\} ||A_{ab}||^2 + 16 \sum_a (\lambda_a^a)^2 ||A_{aa}||^2,$$

$$L_N^a = 8 \sum_{a \neq b} \lambda_a^a \lambda_b^b \langle A_{aa}, A_{bb} \rangle + 8 \sum_a (\lambda_a^a)^2 ||A_{aa}||^2,$$

and so

$$K_N^a + L_N^a = 8 \sum_{a \neq b} \left[ \{(\lambda_a^a)^2 + (\lambda_b^b)^2\} ||A_{ab}||^2 + \lambda_a^a \lambda_b^b \langle A_{aa}, A_{bb} \rangle \right]$$

$$+ 24 \sum_a (\lambda_a^a)^2 ||A_{aa}||^2.$$ (2.9)

Since we have the following inequalities

$$2\lambda_a^a \lambda_b^b \langle A_{aa}, A_{bb} \rangle \leq -\{(\lambda_a^a)^2 + (\lambda_b^b)^2\} \langle A_{aa}, A_{bb} \rangle \quad \text{if} \quad \langle A_{aa}, A_{bb} \rangle < 0,$$

$$2\lambda_a^a \lambda_b^b \langle A_{aa}, A_{bb} \rangle \leq \{(\lambda_a^a)^2 + (\lambda_b^b)^2\} \langle A_{aa}, A_{bb} \rangle \quad \text{if} \quad \langle A_{aa}, A_{bb} \rangle > 0,$$

and $\langle A_{aa}, A_{bb} \rangle < ||A_{aa}|| \cdot ||A_{bb}||$, we get

$$2\lambda_a^a \lambda_b^b \langle A_{aa}, A_{bb} \rangle \leq \{(\lambda_a^a)^2 + (\lambda_b^b)^2\} ||A_{aa}|| \cdot ||A_{bb}||.$$ (2.10)

It follows from (2.9) and (2.10) that we have

$$K_N^a + L_N^a \leq 4 \sum_{a \neq b} \{(\lambda_a^a)^2 + (\lambda_b^b)^2\} \left\{ 2||A_{ab}||^2 + ||A_{aa}|| \cdot ||A_{bb}|| \right\}$$

$$+ 24 \sum_a (\lambda_a^a)^2 ||A_{aa}||^2.$$ (2.11)

Hence, using (2.3) and (2.6), from (2.11) we obtain

$$K_N^a + L_N^a \leq 4(c - c) \left[ \sum_{a \neq b} \{(\lambda_a^a)^2 + (\lambda_b^b)^2\} + 3 \sum_a (\lambda_a^a)^2 \right]$$

$$= 4(2n + 1)(c - c) \sum_a (\lambda_a^a)^2$$

$$= (2n + 1)(c - c) (\text{Tr} H_a^a + \text{Tr} H_{a^*}^a).$$ (2.12)

Since $K_N + L_N = \Sigma_a (K_N^a + L_N^a)$ and $\|\sigma\|^2 = \Sigma_a (\text{Tr} H_a^a + \text{Tr} H_{a^*}^a)$, (2.12) implies (2.7). Q.E.D.

Proof of Theorem. Using (2.7), from (1.6) we have

$$\frac{1}{2} \Delta \|\sigma\|^2 > \|\nabla \sigma\|^2 + \frac{n + 2}{2} c \|\sigma\|^2 - (2n + 1)(c - c) \|\sigma\|^2$$

$$> (2n + 1) \left( c - \frac{3n \bar{c}}{4n + 2} \right) \|\sigma\|^2 \quad \text{on} \ M.$$ (2.13)
Since $M$ is compact and $c > 3n\tilde{c}/(4n + 2)$, from (2.13) $\Delta||\sigma||^2 = 0$ on $M$. Hence we have $||\sigma|| = 0$ identically on $M$, so that $M$ is totally geodesic. Q.E.D.

**Bibliography**


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