LONGITUDES OF A LINK AND PRINCIPALITY
OF AN ALEXANDER IDEAL

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ABSTRACT. In this note it is shown that the longitudes of a \( \mu \)-component homology boundary link \( L \) are in the second commutator subgroup \( G'' \) of the link group \( G \) if and only if the \( \mu \)th Alexander ideal \( \mathcal{A}_\mu(L) \) is principal, generalizing the result announced for \( \mu = 2 \) by R. H. Crowell and E. H. Brown. These two properties were separately hypothesized as characterizations of boundary links by R. H. Fox and N. F. Smythe.

For a \( \mu \)-component homology boundary link \( L \) the first nonvanishing Alexander ideal is \( \mathcal{A}_\mu(L) \). If \( L \) is actually a boundary link, then \( \mathcal{A}_\mu(L) \) is principal and the longitudes of \( L \) lie in the second commutator subgroup of the link group \([2],[6]\). R. H. Crowell and E. H. Brown have announced that the latter two assertions are equivalent for a 2-component homology boundary link \([2]\). This note presents a proof of the following generalization.

**Theorem.** Let \( L: \bigcup_{j=1}^\mu S^1 \to S^3 \) be a (locally flat) \( \mu \)-component homology boundary link, with group \( G \). Then \( \mathcal{A}_\mu(L) = (\Delta_\mu) \cdot A \) where \( A \) is contained in the annihilator ideal (in \( \Lambda = \mathbb{Z}[\mathbb{Z}^\mu] \approx \mathbb{Z}[t_1, t_1^{-1}, \ldots, t_\mu, t_\mu^{-1}] \) of the image of the longitudes in the \( \Lambda \)-module \( G'/G'' \)), and \( A \) is contained in no proper principal ideal. Hence \( \mathcal{A}_\mu(L) \) is principal if and only if the longitudes of \( L \) lie in \( G'' \).

**Proof.** \( L \) extends to an imbedding \( N: \bigcup_{j=1}^\mu S^1 \times D^2 \to S^3 \), since it is locally flat. Let \( X = S^3 - \text{int}(\text{Im}(N)) \) have base point \( x_0 \in X - \partial X \). Then \( G \approx \pi_1(X, x_0) \). Let \( p: X' \to X \) be the maximal abelian cover of \( X \) and choose \( x'_0 \in p^{-1}(x_0) \), so that \( \pi_1(X', x'_0) \approx G' \) and \( H_1(X') = G'/G'' \). By definition of homology boundary link there is a map

\[
\begin{align*}
    f: (X, x_0) &\to \bigvee_{j=1}^\mu S^1, \\
    (x, x_0) &\to \bullet
\end{align*}
\]

inducing an epimorphism of fundamental groups, and \( p \) is the pullback via \( f \) of the maximal abelian cover of \( \bigvee_{j=1}^\mu S^1 \). Thus \( X' \) may be constructed by splitting \( X \) along "Seifert surfaces", as was done in \([3]\) for boundary links. For
each \( j \) such that \( 1 < j < \mu \), choose \( P_j \in S^j \) distinct from the wedge-point \( * \), and let \( V_j = f^{-1}(P_j) \). After homotoping \( f \) if necessary, each \( V_j \) may be assumed a connected, bicolllared submanifold. Let \( Y = X - \bigcup_{j=1}^{\mu} \text{int} W_j \) where the \( W_j \) are disjoint regular neighborhoods of the \( V_j \) in \( X \). There are two natural embeddings of each \( V_j \) in \( Y \); call one \( \nu_{j+} \) and the other \( \nu_{j-} \). (Making such a choice is equivalent to choosing a local orientation for each \( P_j \) in \( \bigcup_{j=1}^{\mu} S^j \), or choosing orientations for the meridians of \( L \).) \( Y \) is a deformation retract of \( X - V \), where \( V = \bigcup_{j=1}^{\mu} V_j \). Then one has

\[
X' = Y \times \mathbb{Z} / \nu_{j+}(w) \times \langle n_1, \ldots, n_j + 1, \ldots, n_\mu \rangle
\]

\[
\sim \nu_{j-}(w) \times \langle \hat{n}_1, \ldots, n_j, \ldots, n_\mu \rangle, \quad \forall w \in V_j, \quad 1 < j < \mu.
\]

\( G'/G'' = H_1(X') \) then appears in the following segment of a Mayer-Vietoris sequence:

\[
\begin{align*}
\cdots & \xrightarrow{d_0} H_0(V) \otimes \Lambda \xrightarrow{d_1} H_1(V) \otimes \Lambda \xrightarrow{\alpha} H_1(X') \\
& \xrightarrow{\delta} H_0(Y) \otimes \Lambda \xrightarrow{d_0} H_0(X') \otimes \Lambda \xrightarrow{\alpha} \mathbb{Z} \xrightarrow{0}
\end{align*}
\]

where \( d_1|H_*(V) \otimes \Lambda = (\nu_{j+})_* \otimes t_j - (\nu_{j-})_* \otimes 1 \) and homology is taken with integral coefficients. The map \( f \) induces a map from this Mayer-Vietoris sequence to the corresponding one for the maximal abelian covering space of \( \bigcup_{j=1}^{\mu} S^j \):

\[
0 - F(\mu)/F(\mu)'' \xrightarrow{\delta} \Lambda^\mu \xrightarrow{\epsilon} \mathbb{Z} \xrightarrow{0}.
\]

(Here \( F(\mu) \) is the free group of rank \( \mu \), and \( \epsilon: \Lambda \rightarrow \mathbb{Z} \) is the augmentation homomorphism.) Since each \( V_j \) is connected, the maps on the degree zero terms are all isomorphisms. Thus one concludes that

\[
H_1(V) \otimes \Lambda \xrightarrow{d_1} H_1(Y) \otimes \Lambda \xrightarrow{\alpha} K \rightarrow 0
\]

is exact, where

\[
K = \ker(\phi: G'/G'' \rightarrow F(\mu)/F(\mu)'' = \ker(\phi: H_1(X') \rightarrow H_0(V) \otimes \Lambda).
\]

Likewise \( f \) induces a map from the 4 term exact sequence of Crowell [1]

\[
0 \rightarrow G'/G'' \rightarrow A(G) \rightarrow \Lambda \xrightarrow{\epsilon} \mathbb{Z} \xrightarrow{0}
\]

to the corresponding sequence for \( F(\mu) \) (which is just the above Mayer-Vietoris sequence for \( \bigcup_{j=1}^{\mu} S^j \)) and so

\[
0 - K \rightarrow A(G) \rightarrow A(F(\mu)) = \Lambda^\mu \rightarrow 0
\]

is exact. From this last short exact sequence one concludes that \( \delta_k(L) = \delta_k(A(G)) \) is equal to the ideal generated by \( \bigcup_{j=0}^{k} \delta_j(K) \cdot \delta_{k-j}(\Lambda^\mu) \); in particular \( \delta_{\mu-1}(L) = 0 \) and \( \delta_\mu(L) = \delta_0(K) \).

Now the \( \Lambda \)-submodule of \( H_1(X') \) generated by the longitudes is the image of \( H_1(\partial X') \) via the inclusion map, and is contained in the image of \( H_1(Y) \otimes \Lambda \), so is contained in \( K \). Let \( B \) be this submodule, and let \( Q \) be the quotient \( \Lambda \)-module. Thus \( 0 - B \rightarrow Q \rightarrow 0 \) is exact, and \( \delta_0(K) = \delta_0(Q) \cdot \delta_0(B) \) (because \( Q \) has a square presentation matrix—see below). It is easy to see that \((\text{Ann}(B))^\mu \subset \delta_0(B)\): if
is a presentation for $B$ with $\varphi(e_i) = \text{eth longitude (where } e_i \text{ is the } i\text{th standard basis element of } \Lambda^n\text{), and if } \alpha_1, \ldots, \alpha_\mu \in \text{Ann}(B) \text{ then}

\Lambda^n \otimes \Lambda^n \rightarrow \Lambda^n \otimes \bar{M} \rightarrow B \rightarrow 0

is also a presentation for $B$, where $\bar{M} = (M, \text{diag}(\alpha_1, \ldots, \alpha_\mu))$, and so

$$\prod_{i=1}^\mu \alpha_i = \det(\text{diag}(\alpha_1, \ldots, \alpha_\mu)) \in \mathfrak{S}_0(B).$$

It is scarcely more difficult to see that $\mathfrak{S}_0(B) \subset \text{Ann}(B)$: let $\delta$ be the determinant of the $\mu \times \mu$ minor $M''$ of $M$. Then

$$\Lambda^n \rightarrow \Lambda^n \rightarrow \text{Coker } M'' \rightarrow 0$$

presents a module of which $B$ is a quotient. Now if $\sum m_i e_i \in \Lambda^n$, then by Cramer's rule $\delta \cdot \sum m_i e_i = M''(\sum n_j e_j)$ where $n_j$ is the determinant at the matrix obtained by replacing the $i$th column of $M''$ with the column of coefficients $(m_i)$. Hence $\delta$ annihilates $\text{Coker } M''$, and a fortiori, $B$. Therefore $\mathfrak{S}_0(B)$, which is generated by such determinants, is contained in $\text{Ann}(B)$. Thus to prove the theorem it will suffice to show that $\mathfrak{S}_0(B)$ is not contained in any proper principal ideal, and that $Q$ has a presentation of the form $\Lambda^q \rightarrow \Lambda^q \rightarrow Q \rightarrow 0$ so that $\mathfrak{S}_0(Q) = (\det P)$ is principal.

Choose base points in $V_i \cap \partial N(S^1_i \times D^2)$ for each $i$, $1 \leq i \leq \mu$, and choose paths from these base points to $\alpha_0$. (Equivalently, $X'$ contains copies of $V_i$ indexed by $\mathbb{Z}^\mu$. Choose one such lift, $V'_i$, for each $i$.) If one now orients the link $L$, the longitudes are unambiguously defined, as elements of $G$. Let $l_i$ be the image of the $i$th longitude in $B$. Since the $i$th longitude commutes with the $i$th meridian, one has $(t_i - 1)l_i = 0$. In contrast to the case of boundary links, $\partial V'_i$ will in general have several components; however $\partial V'_i \cap \partial N(S^1_i \times D^2)$ is always homologous in $\partial N(S^1_i \times D^2)$ to the $i$th longitude, if $i = j$, and to 0 otherwise. $\partial V'_i$ is a union of translates of loops in the homology classes $l_1, \ldots, l_\mu$. Hence there are relations of the form

$$\sum_{j=1}^\mu p_{ij}(t_1, \ldots, t_\mu)l_j = 0$$

in $B$, and by the above remarks on $\partial V_j$, one has $p_{ij}(1, \ldots, 1, 1) = 0$ for $i \neq j$ and $p_{ii}(1, \ldots, 1, 1) = \pm 1$. Since $t_i \cdot l_i = 1 \cdot l_i$, one may assume that $p_i = p_0(t_1, \ldots, t_\mu)$ does not involve $l_i$. Clearly $p_i \prod_{j \neq i} (t_j - 1)$ is the determinant of a $\mu \times \mu$ matrix of relations for $B$, and so is in $\mathfrak{S}_0(B)$. (For what follows it would be sufficient to observe that it clearly annihilates $B$, and so the $\mu$th power is in $\mathfrak{S}_0(B)$.) Let $(c)$ be a principal ideal containing $\mathfrak{S}_0(B)$. Since $\Lambda$ is a factorial domain, $c$ may be assumed irreducible. Therefore $p_i \prod_{j \neq i} (t_j - 1) \in (c)$ implies $c$ divides $p_i$, or some $(t_j - 1)$ for $j > 1$. If $c = t_j - 1$, then $c$ cannot divide $p_i \prod_{k \neq j} (t_k - 1)$ which does not involve $t_j$. If $c$ divides $p_i$ for each $i,$
1 \leq i \leq \mu$, then $c$ involves none of the variables and hence is in $\mathbb{Z}$. Since $p_i(1, \ldots, 1) = \pm 1$, $c = \pm 1$ and so $(c) = \Lambda$.

Let $J = \ker(H_1(X - V, \partial X - V) \to H_0(\partial X - V)) = H_1(X - V)/H_1(\partial X - V)$. From the following commutative diagram of $\Lambda$-modules

$$
\begin{array}{ccc}
H_1(\partial V) \otimes \Lambda & \to & H_1(V) \otimes \Lambda \\
\downarrow & & \downarrow \\
H_1(\partial X - V) \otimes \Lambda & \to & H_1(X - V) \otimes \Lambda \\
\downarrow & & \downarrow \\
H_1(\partial X') & \to & H_1(X') \\
\end{array}
$$

(with rows from exact sequences of pairs and columns from Mayer-Vietoris sequences of $\mathbb{Z}^n$-covers), one deduces a commutative diagram

$$
\begin{array}{ccc}
H_1(\partial V) \otimes \Lambda & \to & H_1(V) \otimes \Lambda \\
\downarrow & & \downarrow \\
H_1(\partial X - V) \otimes \Lambda & \to & H_1(X - V) \otimes \Lambda \\
\downarrow & & \downarrow \\
0 & \to & 0 \\
\end{array}
$$

and

$$
\begin{array}{ccc}
K & \to & Q \\
\downarrow & & \downarrow \\
0 & \to & 0 \\
\end{array}
$$

in which all rows and the first two columns are exact. It follows that the third column is exact, and so

$$
(H_1(V)/H_1(\partial V)) \otimes \Lambda \to J \otimes \Lambda \to Q \to 0
$$

is a presentation for $Q$. Let $\rho = \text{rk}_Z H_1(V)$, $\sigma = \text{rk}_Z H_1(\partial V)$. Since $0 \to H_2(V, \partial V) \to H_1(\partial V) \to H_1(V)$ is exact, one has $\text{rk}_Z(H_1(V)/H_1(\partial V)) = \rho - \sigma + \mu$. Similarly,

$$
H_1(X - V, \partial X - V) \to H_0(\partial X - V) \to H_0(X - V) \to 0
$$

is exact, and $\text{rk}_Z H_0(\partial X - V) = \sigma$, $\text{rk}_Z H_0(X - V) = 1$, so

$$
\text{rk}_Z J = \text{rk}_Z H_1(X - V, \partial X - V) - \sigma + 1 = \text{rk}_Z H_1(S^3 - V, \text{Im } N) - \sigma + 1.
$$

Now each component of the link is the homology boundary of a (singular) surface in $S^3 - V$, and so the natural map

$$
H_1(\text{Im } N) \to H_1(S^3 - V)
$$

is null. Therefore

$$
0 - H_1(S^3 - V) \to H_1(S^3 - V, \text{Im } N) \to H_0(\text{Im } N) \to H_0(S^3 - V) \to 0
$$

is exact, and so $\text{rk}_Z H_1(S^3 - V, \text{Im } N) = \text{rk}_Z H_1(S^3 - V) + \mu - 1 = \text{rk}_Z H_1(V) + \mu - 1$ by Alexander duality $= \rho + \mu - 1$. Thus $\text{rk}_Z J = \rho + \mu$. 


\[ -\sigma = \text{rk}_2(H_1(V)/H_1(\partial V)), \]
and so \( \mathcal{E}_0(Q) \) is principal. This completes the proof of the theorem.

\textbf{Remarks.} 1. Brown and Crowell asserted the somewhat more precise result (for \( \mu = 2 \)) that \( A \) could be generated by 3 elements, of the form \((t_1 - 1)p_1(t_1), (t_2 - 1)p_2(t_2) \) and \( p_i(t_i) + p_2(t_2) - 1 \) where \( p_i(1) = 1 \), and that the \( i \)th longitude lay in \( G'' \) if and only if \( p_{3-i}(t_{3-i}) \) were a unit [2]. This follows readily from \( A = A_1 \cap A_2 \), where \( A_i \) is the annihilator of the \( i \)th longitude and equals \((t_i - 1, p_{3-i}(t_{3-i})) \) for some \( p_i \), as above.

2. Fox and Smythe conjectured that if the longitudes were in \( G'' \), then the link would be a boundary link [6]. H. W. Lambert has constructed a 2-component homology boundary link which is not a boundary link, as a counterexample to this conjecture [4]. (Figure 1 of his paper is incorrectly drawn: the shorter longitude of this example does not map to 0 in the Alexander module (via Crowell’s inclusion \( 0 \rightarrow G'/G'' \rightarrow A(G) \) [1]) and hence this link is not such a counterexample.\(^1\)) Notice also that boundary links have the stronger (but less tractable?) property that the longitudes are in \((G_\omega')\) (where \( G_\omega = \bigcap_{n=1}^{\infty} G_n \) is the intersection of the terms of the lower central series). This follows from the construction of the \( \omega \)-covering by splitting the link complement along Seifert surfaces, as in [3].

3. If \( L \) is trivial then \( \mathcal{E}_k(L) = \Lambda \), but the converse is false, even for knots (\( \mu = 1 \)), for there exists nontrivial knots (for instance doubled knots with twist number 0) with Alexander polynomial 1 [5].

\textbf{References}


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\(^1\)Lambert has advised me that his argument is based on a slightly different figure.