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A STRONGER BORSUK-ULAM TYPE THEOREM FOR PROPER Z_p-ACTIONS ON MOD p HOMOLOGY n-SPHERES

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ABSTRACT. We obtain under a less restrictive hypothesis a Borsuk-Ulam type result of Munkholm's in which the antipodal map is replaced by a Z_p-action on a cohomology n-sphere.

In 1960 Conner and Floyd [2] and [3] proved the following generalization of the Borsuk-Ulam and Bourgin-Yang theorems:

THEOREM (CONNER AND FLOYD). *If t is a differentiable involution on the n-sphere S^n and f is any map of S^n into a differentiable k-manifold M with the property that f_*: $H_n(S^n, Z_2) \to H_n(M, Z_2)$ is trivial, then the set of points x with $f(x) = f(t(x))$ has covering dimension at least $n - k$.*

In [2] Conner and Floyd asked if all differentiability hypotheses could be eliminated, and if S^n might be replaced by a closed, topological manifold which is a mod 2 homology n-sphere. In 1969 Munkholm [5] gave an affirmative answer to the first question. In a later paper [6] Munkholm proved the following theorem, thereby showing that under an additional restriction on f the second question of Conner and Floyd's has a positive answer.

THEOREM (MUNKHOLM). *Let t be a fixed-point free map of prime period p on a closed, topological manifold V which is a* mod *p homology n-sphere. Let f be a nice map of V into a compact, topological k-manifold M, where f is said to be nice provided there is a point $y \in M$ and a map f_0, homotopic to f, which maps at most one point of each orbit in V to a point other than y. In the case $p = 2$ assume that*

$$f_*: H_n(V, Z_2) \to H_n(M, Z_2)$$

is trivial, and if $p \neq 2$ assume M is Z_p-orientable. Then the covering dimension of the set of points x with

$$f(x) = f(t(x)) = \cdots = f(t^{p-1}(x))$$

is at least $n - k(p - 1)$.

In this note the theorem of Munkholm is proved without the restriction that f be nice. The method of proof is to show that under the hypothesis a

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certain cohomology class in $H^{k(p-1)}(E_{z_p} \times_{z_p} M^p)$, which was defined by Haefliger in [4], has nonzero image in $H^{k(p-1)}(V/Z_p)$ under a map

$$f_\pi^*: V/Z_p \to E_{z_p} \times_{z_p} M^p$$

which is induced by f. In §1 we give the definition of the Haefliger class and Haefliger's explicit formulation of the class in terms of elements of $H^*(Z_p)$ and of the characteristic classes of Wu.

We remark as in [6] that to show the covering dimension of A is at least m, it suffices to show that $\overline{H}_c^m(A) \neq 0$, where $\overline{H}_c$ denotes Alexander-Spanier cohomology with compact supports. By manifold we shall always mean topological manifold.

Another type of generalization of the Borsuk-Ulam theorem to a space with a fixed-point free p-period map t is considered by Connett and Cohen in [9] and by Cohen and Lusk in [10]. They are concerned with the set of points x with $f(x) = f(t^i(x))$ for some integer i, $1 < i < p$.

This paper is derived from the author's thesis at the University of Houston. I would like to express my warmest gratitude to Professor D. G. Bourgin who suggested the problem to me and who pointed out the relevance of the Haefliger class to the problem.

1. The Haefliger class. In [4] Haefliger defined for each prime p and each permutation group π of p objects, a universal class of p-tuple points of type π for mappings into a manifold M. We are interested in the case that π is the cyclic group Z_p and give the definition for that case.

DEFINITION. Let M be a manifold of dimension k, let p be a prime, and let π denote the group of cyclic permutations of p objects. Let $E'_\pi = S^{2N+1}$ for sufficiently large N with standard π action. Let π act on M^p by permutation of the factors and on M by the trivial action. The twisted products $E'_\pi \times_\pi M^p$ and $E'_\pi \times_\pi M$ are manifolds, and $E'_\pi \times_\pi M$ will be considered to be a submanifold of $E'_\pi \times_\pi M^p$ via the diagonal embedding. Let $u'_\pi \in H^{k(p-1)}(E'_\pi \times_\pi M^p)$ be the image under Poincaré duality of the homology class of $E'_\pi \times_\pi M^p$ represented by the submanifold $E'_\pi \times_\pi M$, where the coefficients are assumed to be the integers if M is orientable or Z_2 otherwise. Let $E_\pi \to B_\pi$ be a universal bundle for π. There exists a representation g, unique up to homotopy, of $E'_\pi \times_\pi M^p$ in $E_\pi \times_\pi M^p$ that induces an isomorphism in cohomology for dimensions up through N. Define the Haefliger class u_π to be $g^{*-1}(u'_\pi)$.

Henceforth we shall consider all cohomology groups to have coefficients Z_p. We shall denote u_π reduced mod p by u_π.

Recall that $H^*(B_\pi)$ may be identified with $H^*(\pi)$ and that

$$H^*(\pi) = \begin{cases} Z_p(\mu) \otimes E(\nu), & p > 2, \\ Z_p(\mu), & p = 2, \end{cases}$$

where μ is of degree 2 if $p > 2$ and of degree 1 if $p = 2$, and ν is of degree 1.

Note that since π acts trivially on M,

$$H^*(E_\pi \times_\pi M) = H^*(B_\pi \times M).$$

Note also that if $q\colon E \to B$ is any fiber bundle with structure group π, then $H^*(E)$ may be given the structure of an $H^*(\pi)$-algebra by defining αx to be $q^*(\sigma(\alpha)) \cup x$ for $\alpha \in H^*(\pi)$ and $x \in H^*(E)$, where $\sigma\colon H^*(B_\pi) \to H^*(B)$ is the characteristic map.

In order to give Haefliger's explicit formulation of the class u_π we need the following theorem of Steenrod, cf. [7]. We assume M is compact.

THEOREM (STEENROD). *There is a natural identification of the $H^*(\pi)$-algebras $H^*(E_\pi \times_\pi M^p)$ and $H^*(\pi, H^*(M^p)) = H^*(\pi) \otimes D^* + N^*$, where N^* denotes the image of the norm homomorphism in $H^*(M^p)$ and D^* denotes the submodule of $H^*(M^p)$ consisting of those elements which are fixed by π.*

Suppose M is compact with boundary B, and if $p > 2$ assume M is Z_p-oriented. Let $w \in H_n(M, B; Z_p)$ be the fundamental homology class defined by the orientation of M. For each positive integer j, Wu has defined in [8] a unique cohomology class $U^j_{(p)}$ belonging to $H^{2j(p-1)}(M, Z_p)$ if $p > 2$ or to $H^j(M, Z_2)$ if $p = 2$ such that if α is in $H^*(M, B; Z_p)$ then

$$\langle Sq^j(\alpha), w\rangle = \langle U^j_{(2)} \cup \alpha, w\rangle, \qquad p = 2,$$

$$\langle p^j(\alpha), w\rangle = \langle U^j_{(p)} \cup \alpha, w\rangle, \qquad p > 2,$$

where $\langle\ ,\ \rangle$ denotes the Kronecker pairing, understood to be zero unless the two entries are of the same dimension. We remark that $U^j_{(2)} = 0$ whenever $j > n - j$ since $Sq^j(\alpha) = 0$ whenever $j > \dim \alpha$; similarly, if $p > 2$, $U^j_{(p)} = 0$ whenever $j > n - 2j(p-1)$.

We now give Haefliger's identification of the class u_π, reduced modulo p, where M is compact and p is prime.

THEOREM (HAEFLIGER). *The class u_π in $H^{k(p-1)}(E_\pi \times_\pi M^p; Z_p)$ is*

$$u_\pi = \sum_{j=0}^{[k/2]} \mu^{k-2j}\left(U^j_{(2)}\right)^2 + \delta_2, \qquad p = 2,$$

$$u_\pi = \lambda \sum_{j=0}^{[k/2p]} (-1)^j \mu^{(p-1)(k-2jp)/2}\left(U^j_{(p)}\right)^p + \delta_p, \qquad p > 2,$$

where $\delta_2 \in H^k(M^2, Z_2)$ (respectively $\delta_p \in H^{k(p-1)}(M^p, Z_p)$) is the cohomology class dual to the homology class in $H_k(M^2, Z_2)$ (respectively in $H_k(M^p, Z_p)$) represented by the diagonal, and $\lambda = (-1)^{k/2}$ or $(-1)^{(k-1)/2}((p-1)/2)!$ depending on whether k is even or odd.

For a proof see [4].

2. Throughout this section, assume that M is a closed k-manifold that is Z_p-orientable.

LEMMA 1. *Let j denote the inclusion of $E_\pi \times_\pi (M^p - M)$ into $E_\pi \times_\pi M^p$. Then $j^*(u_\pi) = 0$.*

PROOF. This follows immediately from [4, 5.2].

Assume that V is a Z_p-orientable manifold of dimension n on which π acts freely, and let t be a generator for π. Let h: $V \to E_\pi$ be a representation of V in the universal space E_π for π. Let f be a map of V into M, and denote by f'_π the map from V/π into $V \times_\pi M^p$ defined by

$$f'_\pi[x]_\pi = [x; f(x), f(tx), \dots, f(t^{p-1}x)]_\pi.$$

Define f_π: $V/\pi \to E_\pi \times_\pi M^p$ to be $(h \times_\pi 1) \circ f'_\pi$.

LEMMA 2. *If $f_\pi^*(u_\pi) \neq 0$, the covering dimension of A, where $A = \{x \in V | f(x) = f(t^i x), i = 1, \dots, p-1\}$, is at least $n - k(p-1)$.*

PROOF. Since A is closed and π-invariant, $V - A$ is a manifold on which π acts freely. Denote the restriction of f to $V - A$ by $\tilde{f}$. Since $\tilde{f}_\pi$ factors through $E_\pi \times_\pi (M^p - M)$ we have, according to Lemma 1, $\tilde{f}_\pi^*(u_\pi) = 0$. In the cohomology exact sequence for the pair $(V/\pi, (V - A)/\pi)$, $f_\pi^*(u_\pi)$ maps to $\tilde{f}_\pi^*(u_\pi) = 0$. Hence $f_\pi^*(u_\pi)$ has a nonzero antecedent in $H^{k(p-1)}(V/\pi, (V - A)/\pi)$. Since we have field coefficients there is a corresponding nonzero element in $H_{k(p-1)}(V/\pi, (V - A)/\pi)$. By Alexander duality $\bar{H}_c^{n-k(p-1)}(A/\pi) \neq 0$, and the lemma follows.

Let T: $V \to V^p$ be defined by $T(x) = (x, tx, \dots, t^{p-1}x)$, and let π act on V^p by permutation of the factors without regard to the action of π on V.

LEMMA 3. *The homomorphism f_π^*: $H^*(E_\pi \times_\pi M^p) \to H^*(V/\pi)$ may be identified with*

$$(1 \times_\pi T)^* \circ (1 \times_\pi f^p)^*: H^*(E_\pi \times_\pi M^p) \to H^*(E_\pi \times_\pi V).$$

PROOF. Consider the following diagram:

$$\begin{array}{ccccc}
H^*(E_\pi \times_\pi M^p) & \xrightarrow{(1 \times_\pi f^p)^*} & H^*(E_\pi \times_\pi V^p) & \xrightarrow{(1 \times_\pi T)^*} & H^*(E_\pi \times_\pi V) \\
\big\downarrow (h \times_\pi 1)^* & & \big\downarrow (h \times_\pi 1)^* & & \big\uparrow \eta \\
H^*(V \times_\pi M^p) & \xrightarrow{(1 \times_\pi f^p)^*} & H^*(V \times_\pi V^p) & \xrightarrow{\delta^*} & H^*(V/\pi)
\end{array}$$

where δ is defined by $[x]_\pi \to [x; x, tx, \dots, t^{p-1}x]_\pi$, η is the canonical isomorphism, and h is a representation of V in E_π.

It is clear that $\delta^* \circ (1 \times_\pi f^p)^* \circ (h \times_\pi 1)^* = f_\pi^*$ since $f'_\pi = (1 \times_\pi f^p) \circ \delta$. The left hand square is clearly commutative. Hence it suffices to show commutativity of the right hand square; however, one can see that this follows by noting that the inverse of η is induced by ξ: $V/\pi \to E_\pi \times_\pi V$ where ξ is defined by $\xi([x]_\pi) = [h(x), x]_\pi$.

THEOREM 1. *Suppose that* V *is closed and is a* mod p *homology* n-*sphere*; *and if* $p = 2$, *suppose that* $f_*(H_k(V)) = 0$. *Then, if* $n \geq k(p-1)$, $f_\pi^*(u_\pi) \neq 0$.

PROOF. Assume $n \geq k(p-1)$. Since $(1 \times_\pi f^p)^*$ is an $H^*(\pi)$-homomorphism,

$$(1 \times_\pi f^p)^*(u_\pi) = \begin{cases} \displaystyle\sum_{j=0}^{[k/2]} \mu^{k-2j}\left(f^* U_{(2)}^j\right)^2 + f^{2*}(\delta_2), & p = 2, \\ \displaystyle\lambda \sum_{j=0}^{[k/2p]} \mu^{(p-1)(k-2jp)/2}\left(f^* U_{(p)}^j\right)^p + f^{p*}(\delta_p), & p > 2. \end{cases}$$

A simple but crucial observation, depending on a dimension argument, is that $f^*(U_{(p)}^j) = 0$ if $j \neq 0$ whether $p = 2$ or $p > 2$. Since $U_{(p)}^0$ is the unit cocycle in $H^0(M)$, $f^*(U_{(p)}^0)$ is the unit cocycle in $H^0(V)$. Hence the above equations reduce to

$$(1 \times_\pi f^p)^*(u_\pi) = \begin{cases} \mu^k(1 \otimes 1) + f^{2*}\delta_2, & p = 2, \\ \lambda\mu^{k(p-1)/2}(1 \otimes \cdots \otimes 1) + f^{p*}\delta_p, & p > 2. \end{cases}$$

For each nonzero element $\alpha_1 \otimes \cdots \otimes \alpha_p$ of

$$H_{k(p-1)}(V^p) = \sum_{i_1 + \cdots + i_p = k(p-1)} H_{i_1}(V) \otimes \cdots \otimes H_{i_p}(V),$$

we have $\langle f^{p*}\delta_p, \alpha_1 \otimes \cdots \otimes \alpha_p\rangle = \langle \delta_p, f_*^p(\alpha_1 \otimes \cdots \otimes \alpha_p)\rangle = \langle \delta_p, f_*\alpha_1 \otimes \cdots \otimes f_*\alpha_p\rangle$. However $H_i(V) = 0$ for $0 < i < k(p-1)$. Hence if the above element is not zero, there is an integer j, $1 \leq j \leq p$, such that α_{i_j} belongs to $H_{k(p-1)}(V)$. Thus $f_*\alpha_{i_j} = 0$, by assumption when $p = 2$ and because $H_{k(p-1)}(V) = 0$ when $p > 2$. We conclude that $f^{p*}\delta_p = 0$ whether $p = 2$ or $p > 2$.

Observe that there is a map of fiber bundles

$$\begin{tikzcd} V \arrow[r] \arrow[d] & V^p \arrow[d] \\ E_\pi \times_\pi V \arrow[r, "1 \times_\pi T"] \arrow[d] & E_\pi \times_\pi V^p \arrow[d] \\ B_\pi \arrow[r] & B_\pi \end{tikzcd}$$

Since $(1 \times_\pi f^p)^*(u_\pi)$ is essentially a power of μ, it comes from $H^*(B_\pi)$. Thus, by naturality, $(1 \times_\pi T)^*((1 \times_\pi f^p)^*(u_\pi))$ comes from $H^*(B_\pi)$. But in the relevant degree the map

$$H^*(B_\pi) \to H^*(E_\pi \times_\pi V)$$

is monic so $((1 \times_\pi T)^* \circ (1 \times_\pi f^p)^*)(u_\pi) \neq 0$ as desired.

Combining the above theorem and Lemma 2 with the observation, as in **[6]**, that any map into a compact manifold M may be considered to be a map into the double of M which is closed, we obtain the following theorem.

THEOREM 2. *If V is a closed, topological n-manifold that is a* mod p *homology n-sphere, t is a fixed-point free map on V of prime period p, M is a compact, Z_p-orientable, topological k-manifold, and f is any map of V into M with the property that*

$$f_*\colon H_n(V, Z_p) \to H_n(M, Z_p)$$

is trivial; then the covering dimension of the set of points x with

$$f(x) = f(t(x)) = \cdots = f(t^{p-1}(x))$$

is at least $n - k(p - 1)$.

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