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OPEN BOOK DECOMPOSITIONS OF 3-MANIFOLDS

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ABSTRACT. We prove that every closed, orientable 3-manifold has an open book decomposition with connected binding. We then give some applications of this result.

1. Introduction. A closed n-manifold has an *open book decomposition* if it can be constructed as follows: Let F be a compact $(n-1)$-manifold with $\partial F \neq \emptyset$. Let h be an autohomeomorphism of F which is the identity on ∂F. Take $F \times [0, 1]$ and identify $(h(x), 0)$ with $(x, 1)$ for $x \in F$ and $(y, 0)$ with (y, t) for $y \in \partial F$, $t \in [0, 1]$. For a manifold M so constructed let $q: F \times [0, 1] \to M$ be the quotient map. $q(\partial F)$ is called the *binding*, the $q(F \times \{t\})$ are called the *pages* of the decomposition.

Alexander [1] proved that every closed orientable 3-manifold has an open book decomposition. It is implied in his paper, and has been widely assumed, that one can always find a decomposition with connected binding. We use a theorem proved independently by Hilden and Montesinos (stated in §2) to prove the following theorem, which was first announced in [13].

THEOREM 1. *Every closed orientable* 3-*manifold has an open book decomposition with connected binding.*

This result has been obtained independently, using different techniques, by F. González-Acuña [7].

We work throughout in either the PL or smooth category. Our terminology on braids is consistent with standard usage; we give the book by J. Birman [3] as a reference. For information on branched coverings we refer to R. H. Fox [5].

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2. Branched coverings. The following result is widely known. See Alexander [1].

PROPOSITION 1. *Let N be a closed* 3-*manifold having an open book decomposition with binding A. Suppose f: $M \to N$ is a finite sheeted covering space branched over a link L such that $L \cap A = \emptyset$ and L is transverse to the pages.*

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Then M has an open book decomposition with binding $f^{-1}(A)$.

We use the following presentation of a closed orientable 3-manifold M as a branched covering of S^3 to construct an open book decomposition of M with connected binding.

PROPOSITION 2 (HILDEN [8], [9]; MONTESINOS [11], [12]). *Every closed orientable 3-manifold M can be presented as a 3-fold dihedral covering space f: $M \to S^3$ branched over a knot k.*

Let $\mathfrak{S}_3 = \langle x, y\colon x^2 = y^3 = (xy)^2 = 1\rangle$ be the symmetric group on three symbols. Given a closed orientable 3-manifold M, it follows from Proposition 2 that there is a knot k in S^3 and a representation ρ of $\pi_1(S^3 - k)$ onto $\mathfrak{S}_3$ such that M is the completion of the covering space of $S^3 - k$ corresponding to the subgroup $\rho^{-1}(\mathrm{gp}(x))$ of $\pi_1(S^3 - k)$. Note that a meridian of k is represented in $\mathfrak{S}_3$ by an element of order two. If J is an oriented simple closed curve in $S^3 - k$, then $f^{-1}(J)$ is connected if and only if $\rho([J])$ has order three, where $[J]$ is the homotopy class of J in $\pi_1(S^3 - k)$.

According to Alexander [1] (see also Birman [3]), any knot in S^3 can be presented as the closed braid $\hat{\beta}$ corresponding to a geometric braid β on n strings, for some $n \geqslant 2$. Thus let D be a disc and $Q_n = \{q_1, \ldots, q_n\}$ be a set of distinct points in $\operatorname{int} D$. The element $\beta \in B_n$ determines an auto-homeomorphism h of $D - Q_n$ which is the identity on ∂D. So h determines an open book decomposition of S^3 with binding the braid axis A.

If $\hat{\beta}$ presents the knot k, then this decomposition satisfies the conditions of Proposition 1 with respect to f: $M \to S^3$. Thus we get an open book decomposition of M with binding $f^{-1}(A)$. In the next section we prove that the presentation of k as a closed braid $\hat{\beta}$ may be chosen so that the homotopy class of the braid axis A is represented in $\mathfrak{S}_3$ by an element of order three. The theorem will then follow by the above remarks.

3. Proof of Theorem 1. Let M be a closed orientable 3-manifold, with k the knot in S^3, ρ: $\pi_1(S^3 - k) \to \mathfrak{S}_3$ the representation, and f: $M \to S^3$ the branched covering given by Proposition 2. Let $\hat{\beta}$ be a closed braid presentation of k with axis A. In the open book decomposition of S^3 given by β, orient $[0, 1]$ from 1 to 0, and orient D so that the orientation on $D \times [0, 1]$ is that of a right-handed screw. Give A and $\hat{\beta}$ the induced orientations. We denote again by D and Q_n the images of $D \times \{0\}$ and $Q_n \times \{0\}$ in S^3.

Choose a basepoint s_0 on ∂D. Let α_j be a loop in ∂D, based at s_0, which encloses q_j but does not enclose any other point of Q_n. Give α_j the orientation induced from that of D. Let $a_j = [\alpha_j] \in \pi_1(S^3 - \hat{\beta}, s_0)$. The α_j may be chosen so that $[A] = a_1 \cdots a_n$. The a_j generate $\pi_1(S^3 - \hat{\beta}, s_0)$ and are all conjugate. $\rho(a_j) = xy^{\varepsilon_j}$, where $0 \leqslant \varepsilon_j \leqslant 2$. Take a regular projection of $\hat{\beta}$ onto a plane perpendicular to A and compute an over presentation [4] of $\pi_1(S^3 - \hat{\beta}, s_0)$. The generators $a_1, \ldots, a_n$ are as indicated in Figure 1.

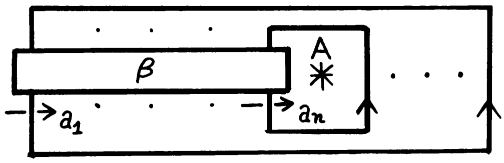

FIGURE 1

Since ρ is onto and $n > 1$, there is an integer m, with $1 \leqslant m \leqslant n-1$, such that $\rho(a_m) \neq \rho(a_{m+1})$. If $n = 2$, then

$$\rho([A]) = xy^{\varepsilon_1}xy^{\varepsilon_2} = y^{\varepsilon_1 - \varepsilon_2} \neq 1;$$

thus $\rho([A])$ has order three and we are done. We may assume that n is even, since if n is odd we can replace β by the braid $\beta\sigma_n \in B_{n+1}$, which also presents k. (The σ_i are the standard Artin generators of the braid group.) So we shall now assume that $n > 2$ and is even.

We may assume that $\rho(a_{n-1}) \neq \rho(a_n)$. If not, there is an integer m as before with $\rho(a_m) \neq \rho(a_{m+1})$. Let

$$\tau_m = (\sigma_m \cdots \sigma_{n-1})(\sigma_{m-1} \cdots \sigma_{n-2})$$

and $\beta' = \tau_m^{-1}\beta\tau_m$. Let A' and a_j' be defined for $\hat{\beta}'$ as A and a_j were defined for $\hat{\beta}$. A projection of $\hat{\beta}'$ is shown in Figure 2. Evidently there is a homeomorphism g of S^3, isotopic to the identity and fixing s_0, such that $g(\hat{\beta}') = \hat{\beta}$ and $g_*(a_{n-1}') = a_{m-1}$, $g_*(a_n') = a_m$, where

$$g_*: \pi_1(S^3 - \hat{\beta}', s_0) \to \pi_1(S^3 - \hat{\beta}, s_0)$$

is induced by $g|(S^3 - \hat{\beta}')$. Let $\rho' = \rho \circ g_*$. Then $\rho'(a_{n-1}') \neq \rho'(a_n')$. So replace $\hat{\beta}$ by the equivalent knot $\hat{\beta}'$ and drop the primes.

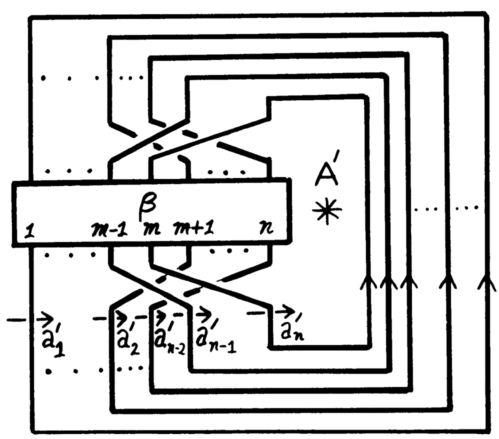

FIGURE 2

Now if $\rho([A]) \neq 1$, then it must have order three, since n is even, and we are done. So assume $\rho([A]) = 1$. We construct a braid $\beta' \in B_{n+2}$ such that $\hat{\beta}'$ is equivalent to $\hat{\beta}$ and has braid axis A' with $\rho'([A']) \neq 1$, where ρ' is the corresponding representation of $\pi_1(S^3 - \hat{\beta}', s_0)$ onto $\mathfrak{S}_3$. Let

$$\beta' = \beta\sigma_n\sigma_{n-1}^{-1}\sigma_n^{-1}\sigma_{n+1}\sigma_n\sigma_{n-1}.$$

A projection of $\hat{\beta}'$ is shown in Figure 3.

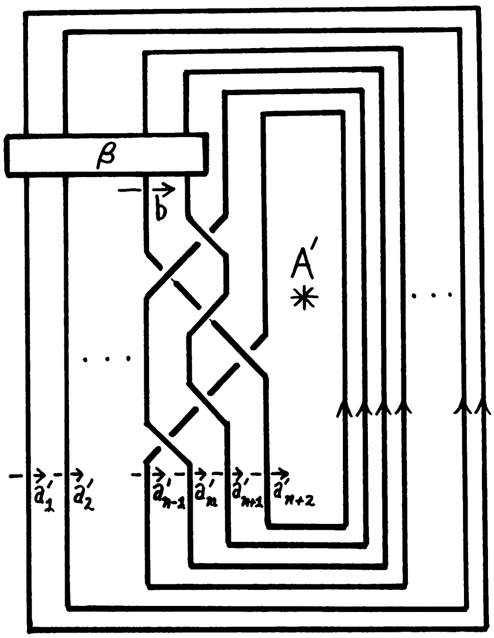

FIGURE 3

There is homeomorphism g from S^3 to itself, isotopic to the identity and fixing s_0, such that $g(\hat{\beta}') = \hat{\beta}'$, $g_*(a_j') = a_j$ for $1 \leqslant j \leqslant n-2$, $g_*(a_{n+1}') = a_n$, and $g_*(b) = a_{n-1}$, where

$$g_*\colon \pi_1(S^3 - \hat{\beta}', s_0) \to \pi_1(S^3 - \hat{\beta}, s_0)$$

is induced by $g|(S^3 - \hat{\beta}')$. From the projection one calculates that

$$g_*(a_n') = g_*\left(a_{n+1}'a_{n+1}'(a_{n+1}')^{-1}\right) = g_*(a_{n+1}') = a_n,$$

$$g_*(a_{n+2}') = g_*\left((a_{n+1}')^{-1}(a_n')^{-1}ba_n'a_{n+1}'\right) = a_n^{-2}a_{n-1}a_n^2,$$

$$\begin{aligned} g_*(a_{n-1}') &= g_*\left(a_n'a_{n+1}'a_{n+2}'a_{n+2}'(a_{n+2}')^{-1}(a_{n+1}')^{-1}(a_n')^{-1}\right) \\ &= g_*\left(a_n'a_{n+1}'a_{n+2}'(a_{n+1}')^{-1}(a_n')^{-1}\right) \\ &= a_na_na_n^{-2}a_{n-1}a_n^2a_n^{-1}a_n^{-1} = a_{n-1}. \end{aligned}$$

Thus $g_*([A']) = g_*(a_1' \cdots a_{n+2}') = a_1 \cdots a_{n-2}a_{n-1}a_n^2a_{n-1} = [A](a_na_{n-1})$. Let $\rho' = \rho \circ g_*$. Then

$$\rho'([A']) = \rho([A])\rho(a_n)\rho(a_{n-1}) = \rho(a_n)\rho(a_{n-1}).$$

Since $(\rho(a_j))^2 = 1$ and $\rho(a_{n-1}) \neq \rho(a_n)$, it follows that $\rho'([A']) \neq 1$, and we are done.

Dropping the primes we see that $f^{-1}(A)$ is the connected binding of the open book decomposition of M lifted from that of S^3 given by β. This completes the proof of Theorem 1.

4. Applications.

COROLLARY 1. *Every closed orientable* 3-*manifold contains fibered knots.*

PROOF. $M - f^{-1}(A)$ is a surface bundle over S^1.

The next result was first proven by R. H. Bing [**2**].

COROLLARY 2. *Every closed orientable* 3-*manifold contains a simple closed curve whose complement is irreducible.*

PROOF. Surface bundles over S^1 whose fibers are not spheres are irreducible.

REMARKS. Note that the method used to prove Corollary 1 is closely related to the method used by D. Goldsmith [**6**] to construct fibered links in S^3. Corollary 2 shows that Theorem 1 yields an alternative proof of Bing's theorem [**2**] that a closed 3-manifold is S^3 if and only if each simple closed curve in M is contained in a 3-cell.

COROLLARY 3. *Every closed orientable* 3-*manifold has a codimension one foliation with precisely one closed leaf*; *this leaf is a torus.*

PROOF. Replace a tubular neighborhood of $f^{-1}(A)$ with a Reeb component and "turbulize" the fibers around it. (See [**10**] for details.)

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