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SHORTER NOTES

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ON ČECH'S THEOREM

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In this note we give an unusual proof of a theorem of Čech [**1**]. Given a completely regular space Y, let βY be the Čech-Stone compactification of Y.

THEOREM. *If X is a metrizable space which is a G_δ in βX, then X is completely metrizable.*

PROOF. For every space Y we let $C^*(Y)$ be the space of continuous bounded real-valued functions on Y, with the usual supremum norm. Note that $\beta f \in C^*(\beta Y)$ if $f \in C^*(Y)$. We will show that X is completely metrizable by finding an embedding ϕ: $X \to C^*(\beta X)$ such that $\phi[X]$ is closed in $C^*(\beta X)$. We may assume that X is not compact. Let ρ be a compatible bounded metric on X. For $s \in X$ define $\rho_s \in C^*(X)$ by $\rho_s(x) = \rho(s, x)$. Note that $\|\rho_s - \rho_t\| = \rho(s, t)$, for $s, t \in X$. Write $\beta X - X$ as $\bigcup_{n \in N} F_n$, with each F_n compact and nonempty. For $n \in N$ define α_n: $X \to R$ by $\alpha_n(s) = \min\{\beta\rho_s(x): x \in F_n\}$. Then $\alpha_n(s) > 0$ for $s \in X$ (for there is an $\varepsilon > 0$ such that $\{x \in X: \rho(s, x) < \varepsilon\}^- \cap F_n = \varnothing$), and each α_n is continuous (for one easily checks that $|\alpha_n(s) - \alpha_n(t)| \leqslant \|\rho_s - \rho_t\|$ whenever $s, t \in X$). It follows that we can define $\phi_n \in C^*(\beta X)$ by

$$\phi_n(s)(x) = \min\{1, \beta\rho_s(x)/\alpha_n(s)\}.$$

Note that $\phi_n(s)(x) = 1$ if $x \in F_n$, for $s \in X$ and $n \in N$. We now claim that the function $\phi = \Sigma\{2^{-n}\phi_n: n \in N\}$ is the required embedding. Evidently ϕ is continuous. If $s, t \in X$ then

$$\begin{aligned}\|\phi(s) - \phi(t)\| &\geqslant |\phi(s)(t) - \phi(t)(t)| \\ &= \phi(s)(t) \geqslant 2^{-1}\min\{1, \rho(s, t)/\alpha_1(s)\}.\end{aligned}$$

It follows that ϕ is one-to-one and that ϕ^{-1} is continuous at $\phi(s)$ for each $s \in X$. It remains to show that $\phi[X]$ is closed. Let $f \in \phi[X]^-$ be arbitrary. There is a sequence $\langle x_k \rangle_k$ in X such that $\langle \phi(x_k) \rangle_k$ converges to f. Let x be a cluster point of $\langle x_k \rangle_k$ in βX. Then $f(x) = 0$ since $\phi(x_k)(x_k) = 0$ for each $k \in N$. Therefore $x \in X$. (For otherwise there is an $n \in N$ with $x \in F_n$, and

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then $f(x) \geqslant 2^{-n}$ since $\phi(x_k)(x) \geqslant 2^{-n}\phi_n(x_k)(x) = 2^{-n}$ for all $k \in N$.) Consequently $f = \phi(x) \in \phi[X]$. $\square$

The initial proof of the theorem was slightly improved by the referee to whom I am very grateful.

References

1. E. Čech, *On bicompact spaces*, Ann. of Math. **38** (1937), 823–844.

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