THE LOCAL HILBERT FUNCTION
OF A PAIR OF PLANE CURVES

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Abstract. Let R be the local ring of a pair of plane curves at a point. In this paper it is proved that the Hilbert function of such a ring changes by at most one at each stage, and it is essentially nonincreasing.

1. Introduction. Let \( \mathfrak{O} \) be an equicharacteristic two dimensional regular local ring and \( f \) and \( g \) be two elements of \( \mathfrak{O} \) which generate an ideal \( \mathfrak{U} \) primary for the maximal ideal of \( \mathfrak{O} \). Let \( R, \quad \text{Gr}(\mathfrak{O}) \) and \( \text{Gr}(R) \) denote respectively the residue-class ring \( \mathfrak{O}/\mathfrak{U} \) and the graded rings of \( \mathfrak{O} \) and \( R \). \( \text{Gr}^i(R) \) denotes the \( i \)th homogeneous component of \( \text{Gr}(R) \) and \( H_i(R) \) denotes its vectorspace dimension over \( \text{Gr}^0(R) \). The function \( H_i(R) \) is referred to as the Hilbert function of \( R \). The orders of \( f \) and \( g \) are denoted by integers \( n \) and \( m \) respectively.

The following questions were raised by Professor Abhyankar.

Question 1. Is \( \sum_{i=0}^{n} H_i(R) > mn \) when \( f \) and \( g \) are tangential, i.e., when their initial forms have a common factor?

Question 2. Is \( |H_{i+1}(R) - H_i(R)| < 1 \) for all nonnegative integers \( i \)?

These questions are answered in the affirmative and the following theorem is proved.

Theorem. \( 0 < H_i(R) - H_{i+1}(R) < 1 \) for \( i > \min(n, m) \).

We will denote by \( x, y \) a regular system of parameters in \( \mathfrak{O} \) and the residue field of \( \mathfrak{O} \) by \( k \). \( \text{Gr}(\mathfrak{O}) \) will be identified with the polynomial ring \( k[X, Y] \) with \( X \) and \( Y \) denoting initial forms of \( x, y \) respectively. With this identification the initial ideal \( \mathfrak{U} \) of \( \mathfrak{O} \) will be a homogeneous ideal in \( k[X, Y] \). The \( i \)th homogeneous component of \( \mathfrak{U} \) and its dimension over \( k \) are denoted by \( \mathfrak{U}_i \) and \( \dim(\mathfrak{U}_i) \) respectively. \( F \) and \( G \) denote initial forms of \( f \) and \( g \). The definitions of the terms used here can be found in [3].

2. For the purpose of the following lemma \( \mathfrak{O} \) will denote the power series ring \( k[[x, y]] \) and the field \( k \) is assumed to be infinite.

Lemma. Let \( f \) and \( g \) be such that \( m > n, g = xg_1 \) with \( g_1 \) in \( \mathfrak{O} \) and \( X \) does not divide \( F \). Let \( \mathfrak{V} \) be the ideal generated by \( f \) and \( g_1 \) and \( \mathfrak{U} \) be its initial ideal. Then \( \dim(\mathfrak{U}_{i+1}) = \dim(\mathfrak{U}_i) + 1 \) for all \( i > n \).

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Proof. If \( W_1, W_2, \ldots, W_r \) is a basis of \( \mathcal{U}_i \) then \( XW_1, XW_2, \ldots, XW_r \) and \( Y^{i+1-n}F \) are elements in \( \mathcal{U}_{i+1} \) which are linearly independent. Hence it readily follows that \( \dim(\mathcal{U}_{i+1}) \geq \dim(\mathcal{U}_i) + 1 \).

Let \( t_i = y^{i+1-n}f \). We prove that there exist elements \( t_2, t_3, \ldots, t_s \) in \( \mathcal{U}_i \), each divisible by \( x \) and such that their initial forms together with \( Y^{i+1-n}F \) form a basis of \( \mathcal{U}_{i+1} \). First choose elements \( t_2^*, t_3^*, \ldots, t_s^* \) such that their initial forms and \( Y^{i+1-n}F \) form a basis of \( \mathcal{U}_{i+1} \). If \( t_j^* \) is divisible by \( x \) then take \( t_j = t_j^* \), otherwise \( t_j \) is chosen as follows. By Weierstrass preparation theorem,

\[
t_1 = \delta[y^{i+1} + c_1(x)y^i + \cdots + c_{i+1}(x)] \quad \text{and} \quad t_j^* = \eta[y^p + d_1(x)y^{p-1} + \cdots + d_p(x)],
\]

where \( \delta \) and \( \eta \) are units and \( c_1(x), \ldots, c_{i+1}(x), d_1(x), \ldots, d_p(x) \) are power series in \( x \) divisible by \( x \). Since \( m \geq n, p \) will be bigger than or equal to \( i + 1 \). Let \( t_j = t_j^* - \eta(\delta^{-1}y^{p-j}t_i) \). The elements \( t_2, t_3, \ldots, t_s \) chosen this way are clearly divisible by \( x \) and their initial forms together with \( Y^{i+1-n}F \) form a basis of \( \mathcal{U}_{i+1} \).

Let \( t_j = a_jf + bg \) for \( j > 2 \). For \( j > 2, t_j \) is divisible by \( x \), write \( t_j = xw_j \) for \( j > 2 \), also \( g = xg_j \). We get \( xw_j = a_jf + xb_jg_j \) which implies that \( x \) divides \( a_j \) and consequently \( w_j \) belongs to \( \mathcal{V}_i \). The initial forms of \( w_j \) are in \( \mathcal{V}_i \) and linearly independent and hence \( \dim(\mathcal{V}_i) + 1 > \dim(\mathcal{U}_{i+1}) \). This completes the proof of the lemma that \( \dim(\mathcal{U}_{i+1}) = \dim(\mathcal{U}_i) + 1 \) for \( i > n \).

3.

Definition. The initial forms of a pair of generators of \( \mathcal{U} \) are said to be irredundant if none of them is a multiple of the other.

Observe that if a pair of generators of \( \mathcal{U} \) is such that their initial forms are irredundant then the sum of the degrees of the initial forms and the degree of the g.c.d. of the initial forms are two well-defined numbers dependent only on the ideal \( \mathcal{U} \). These numbers will be denoted by \( \alpha(\mathcal{U}), \beta(\mathcal{U}) \) respectively and are used for induction in the proof of the following theorem.

Theorem. \( 0 < H_i(R) - H_{i+1}(R) < 1 \) for all \( i \geq \min(n, m) \).

Proof. The completion \( \hat{R} \) of \( R \) is isomorphic to \( \hat{\mathcal{O}} / \hat{\mathcal{U}} \), where \( \hat{\mathcal{O}} \) is the completion \( \mathcal{O} \) of \( \mathcal{U}_i \) and \( \hat{\mathcal{U}} = \mathcal{U} \hat{\mathcal{O}} \) and also since \( \hat{R} \) and \( \hat{\mathcal{O}} \) have isomorphic graded rings \( H_i(R) = H_i(\hat{R}) \). Hence without loss of generality we assume that the rings \( \mathcal{O} \) and \( R \) are complete. By Cohen’s structure theorem \( \mathcal{O} \) is then isomorphic to the power series ring \( k[[x, y]] \).

Since \( \text{Gr}(R) \cong k[X, Y]/\mathcal{U}_i \) and \( H_i(R) = i + 1 - \dim(\mathcal{U}_i) \), the theorem is equivalent to proving \( 1 < \dim(\mathcal{U}_{i+1}) - \dim(\mathcal{U}_i) < 2 \) for all \( i > \min(n, m) \). The computation of \( \dim(\mathcal{U}_i) \) is fairly straightforward for \( i < \min(n, m) + 1 \) and hence the above inequality easily checked for \( i = \min(n, m) \). In view of this it is sufficient to prove \( 1 < \dim(\mathcal{U}_{i+1}) - \dim(\mathcal{U}_i) < 2 \) for \( i > \min(n, m) + 1 \). We give a proof of this statement assuming that the field \( k \) is infinite.
The proof when $k$ is finite can be reduced to that case by extending $k$ by adjoining an indeterminate.

Let $T_1, T_2, \ldots, T_r$ be a basis over $k$ of $\mathcal{U}_i$ for $i \geq \min(n, m) + 1$. If $T_i$ is an element of the basis least divisible by $X$ then $YT_i$ and $XT_1, XT_2, \ldots, XT_r$ are linearly independent. These elements are in $\mathcal{U}_{i+1}$ and we have that $\dim(\mathcal{U}_{i+1}) - \dim(\mathcal{U}_i) > 1$ for $i \geq \min(n, m) + 1$.

It remains to be proved that $\dim(\mathcal{U}_{i+1}) - \dim(\mathcal{U}_i) < 2$ for $i \geq \min(n, m) + 1$. Without loss of generality we can assume that $n < m$ and that the initial forms of $f$ and $g$ are irredundant. Since the field $k$ is infinite we can, if necessary by making suitable linear transformation in $x$ and $y$, assume that $X$ does not divide the initial form of $f$. Then we show that there exists a $g^*$ divisible by $x$ and such that $f$ and $g^*$ generate $\mathcal{U}$ and their initial forms are irredundant. If $x$ divides $g$ then there is nothing to prove hence assume that $x$ does not divide $g$. By Weierstrass preparation theorem

$$f = \delta \left[ y^n + c_1(x)y^{n-1} + \cdots + c_n(x) \right] \quad \text{and}$$

$$g = \eta \left[ y^p + d_1(x)y^{p-1} + \cdots + d_p(x) \right]$$

where $p$ is some integer bigger than or equal to $m$, $\delta$, $\eta$ are units and $c_1(x), \ldots, c_n(x), d_1(x), \ldots, d_p(x)$ are power series in $x$ divisible by $x$. Then $g^* = g - \eta \delta^{-1}y^{p-n}f$ is as desired. Let $g^* = xg_1$ and $\mathcal{V}$ be the ideal generated by $f$ and $g_1$. In view of the lemma it is enough to prove that $\dim(\mathcal{U}_i) - \dim(\mathcal{U}_{i-1}) < 2$ for $i > n + 1$. Since $\mathcal{V}$ is generated by $f$ and $g_1$, $\beta(\mathcal{V}) < \beta(\mathcal{U})$ and if $\beta(\mathcal{V}) = \beta(\mathcal{U})$ then $\alpha(\mathcal{V}) < \alpha(\mathcal{U})$, hence the proof of the theorem follows by induction.

Now in the following corollaries we answer the questions raised by Professor Abhyankar. If $f$ and $g$ are nontangential, i.e., if their initial forms are coprime then $H_i(R)$ depends only on the orders of $f$ and $g$, and we will denote it by $H_i(n, m)$. It is easily checked that $\sum_{i=0}^{\infty} H_i(n, m) = mn$ and $H_i(n, m) = 0$ for all $i > m + n - 1$.

**Corollary 1.** If $f$ and $g$ are tangential then $\sum_{i=0}^{m+n-1} H_i(R) > mn$.

**Proof.** Let $p > 0$ be the degree of the g.c.d. of the initial forms of $f$ and $g$. It is easily checked that $H_i(R) = H_i(n, m)$ for $0 < i < m + n - p - 1$ and $H_{m+n-p}(R) = H_{m+n-p}(n, m) + 1$. By the above theorem, $H_i(R)$ decreases by at most one at each stage for $i \geq \min(n, m)$ whereas $H_i(n, m)$ decreases by one at each stage for $i > \min(n, m)$ until it becomes zero. It follows that $H_i(R) > H_i(n, m)$ for all $i$ and

$$\sum_{i=0}^{m+n-1} H_i(R) > \sum_{i=0}^{m+n-1} H_i(n, m) = mn.$$

**Corollary 2.** $|H_{i+1}(R) - H_i(R)| < 1$ for all nonnegative integers.

**Proof.** For $0 < i < \min(n, m)$ this is checked by direct calculation and for $i > \min(n, m)$ this follows from the theorem.
We deduce the following result of Bertini cited in [1].

**Corollary 3.** Let \( \alpha \) be the intersection multiplicity of \( f \) and \( g \) and \( e \) be the largest integer such that \( H_i(R) > 0 \) for all \( i < e \) then \( e < \alpha + m + n - mn - 2 \).

**Proof.** Since, as observed in the proof of Corollary 1, \( H_i(R) > H_i(n, m) \) for all \( i \) and \( H_i(n, m) = 0 \) for \( i > m + n - 1 \) hence \( \Sigma_{i=0}^{e} H_i(R) > \Sigma_{i=0}^{n+m-2} H_i(n, m) + (e - m - n + 2) \). But \( \Sigma_{i=0}^{e} H_i(R) = \alpha \), the intersection multiplicity and \( \Sigma_{i=0}^{n+m-2} H_i(n, m) = mn \) hence the result follows.

**References**


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