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THE LOCAL HILBERT FUNCTION OF A PAIR OF PLANE CURVES[1]

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ABSTRACT. Let R be the local ring of a pair of plane curves at a point. In this paper it is proved that the Hilbert function of such a ring changes by at most one at each stage, and it is essentially nonincreasing.

1. Introduction. Let $\mathcal{O}$ be an equicharacteristic two dimensional regular local ring and f and g be two elements of $\mathcal{O}$ which generate an ideal $\mathfrak{A}$ primary for the maximal ideal of $\mathcal{O}$. Let R, $\mathrm{Gr}(\mathcal{O})$ and $\mathrm{Gr}(R)$ denote respectively the residue-class ring $\mathcal{O}/\mathfrak{A}$ and the graded rings of $\mathcal{O}$ and R. $\mathrm{Gr}^i(R)$ denotes the ith homogeneous component of $\mathrm{Gr}(R)$ and $H_i(R)$ denotes its vectorspace dimension over $\mathrm{Gr}^0(R)$. The function $H_i(R)$ is referred to as the Hilbert function of R. The orders of f and g are denoted by integers n and m respectively.

The following questions were raised by Professor Abhyankar.

Question 1. Is $\sum_{i=0}^{m+n-1} H_i(R) > mn$ when f and g are tangential, i.e., when their initial forms have a common factor?

Question 2. Is $|H_{i+1}(R) - H_i(R)| \leqslant 1$ for all nonnegative integers i?

These questions are answered in the affirmative and the following theorem is proved.

THEOREM. $0 \leqslant H_i(R) - H_{i+1}(R) \leqslant 1$ *for* $i \geqslant \min(n, m)$.

We will denote by x, y a regular system of parameters in $\mathcal{O}$ and the residue field of $\mathcal{O}$ by k. $\mathrm{Gr}(\mathcal{O})$ will be identified with the polynomial ring $k[X, Y]$ with X and Y denoting initial forms of x, y respectively. With this identification the initial ideal $\overline{\mathfrak{A}}$ of $\mathfrak{A}$ will be a homogeneous ideal in $k[X, Y]$. The ith homogeneous component of $\overline{\mathfrak{A}}$ and its dimension over k are denoted by $\overline{\mathfrak{A}}_i$ and $\dim(\overline{\mathfrak{A}}_i)$ respectively. F and G denote initial forms of f and g. The definitions of the terms used here can be found in [3].

2. For the purpose of the following lemma $\mathcal{O}$ will denote the power series ring $k[[x, y]]$ and the field k is assumed to be infinite.

LEMMA. *Let f and g be such that $m \geqslant n$, $g = xg_1$ with g_1 in $\mathcal{O}$ and X does not divide F. Let $\mathfrak{V}$ be the ideal generated by f and g_1 and $\overline{\mathfrak{V}}$ be its initial ideal. Then* $\dim(\overline{\mathfrak{A}}_{i+1}) = \dim(\overline{\mathfrak{V}}_i) + 1$ *for all* $i \geqslant n$.

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PROOF. If $W_1, W_2, \ldots, W_r$ is a basis of $\overline{\mathcal{V}}_i$ then $XW_1, XW_2, \ldots, XW_r$ and $Y^{i+1-n}F$ are elements in $\overline{\mathfrak{A}}_{i+1}$ which are linearly independent. Hence it readily follows that $\dim(\overline{\mathfrak{A}}_{i+1}) \geqslant \dim(\overline{\mathcal{V}}_i) + 1$.

Let $t_1 = y^{i+1-n}f$. We prove that there exist elements $t_2, t_3, \ldots, t_s$ in $\mathfrak{A}$, each divisible by x and such that their initial forms together with $Y^{i+1-n}F$ form a basis of $\overline{\mathfrak{A}}_{i+1}$. First choose elements $t_2^*, t_3^*, \ldots, t_s^*$ such that their initial forms and $Y^{i+1-n}F$ form a basis of $\overline{\mathfrak{A}}_{i+1}$. If t_j^* is divisible by x then take $t_j = t_j^*$, otherwise t_j is chosen as follows. By Weierstrass preparation theorem,

$$t_1 = \delta\left[y^{i+1} + c_1(x)y^i + \cdots + c_{i+1}(x)\right] \quad \text{and}$$

$$t_j^* = \eta\left[y^p + d_1(x)y^{p-1} + \cdots + d_p(x)\right],$$

where δ and η are units and $c_1(x), \ldots, c_{i+1}(x), d_1(x), \ldots, d_p(x)$ are power series in x divisible by x. Since $m \geqslant n, p$ will be bigger than or equal to $i + 1$. Let $t_j = t_j^* - \eta\delta^{-1}y^{p-i-1}t_1$. The elements $t_2, t_3, \ldots, t_s$ chosen this way are clearly divisible by x and their initial forms together with $Y^{i+1-n}F$ form a basis of $\overline{\mathfrak{A}}_{i+1}$.

Let $t_j = a_jf + b_jg$ for $j \geqslant 2$. For $j \geqslant 2$, t_j is divisible by x, write $t_j = xw_j$ for $j \geqslant 2$, also $g = xg_1$. We get $xw_j = a_jf + xb_jg_1$ which implies that x divides a_j and consequently w_j belongs to $\mathcal{V}$. The initial forms of w_j are in $\overline{\mathcal{V}}_i$ and linearly independent and hence $\dim(\overline{\mathcal{V}}_i) + 1 \geqslant \dim(\overline{\mathfrak{A}}_{i+1})$. This completes the proof of the lemma that $\dim(\overline{\mathfrak{A}}_{i+1}) = \dim(\overline{\mathcal{V}}_i) + 1$ for $i \geqslant n$.

3.

DEFINITION. The initial forms of a pair of generators of $\mathfrak{A}$ are said to be irredundant if none of them is a multiple of the other.

Observe that if a pair of generators of $\mathfrak{A}$ is such that their initial forms are irredundant then the sum of the degrees of the initial forms and the degree of the g.c.d. of the initial forms are two well-defined numbers dependent only on the ideal $\mathfrak{A}$. These numbers will be denoted by $\alpha(\mathfrak{A})$, $\beta(\mathfrak{A})$ respectively and are used for induction in the proof of the following theorem.

THEOREM. $0 \leqslant H_i(R) - H_{i+1}(R) \leqslant 1$ *for all* $i \geqslant \min(n, m)$.

PROOF. The completion $\hat{R}$ of R is isomorphic to $\hat{\mathcal{O}}/\hat{\mathfrak{A}}$, where $\hat{\mathcal{O}}$ is the completion $\mathcal{O}$ and $\hat{\mathfrak{A}} = \mathfrak{A}\hat{\mathcal{O}}$ and also since R and $\hat{R}$ have isomorphic graded rings $H_i(R) = H_i(\hat{R})$. Hence without loss of generality we assume that the rings $\mathcal{O}$ and R are complete. By Cohen's structure theorem $\mathcal{O}$ is then isomorphic to the power series ring $k[[x, y]]$.

Since $\mathrm{Gr}(R) \simeq k[X, Y]/\overline{\mathfrak{A}}$ and $H_i(R) = i + 1 - \dim(\overline{\mathfrak{A}}_i)$, the theorem is equivalent to proving $1 \leqslant \dim(\overline{\mathfrak{A}}_{i+1}) - \dim(\overline{\mathfrak{A}}_i) \leqslant 2$ for all $i \geqslant \min(n, m)$. The computation of $\dim(\overline{\mathfrak{A}}_i)$ is fairly straightforward for $i < \min(n, m) + 1$ and hence the above inequality easily checked for $i = \min(n, m)$. In view of this it is sufficient to prove $1 \leqslant \dim(\overline{\mathfrak{A}}_{i+1}) - \dim(\overline{\mathfrak{A}}_i) < 2$ for $i > \min(n, m) + 1$. We give a proof of this statement assuming that the field k is infinite.

The proof when k is finite can be reduced to that case by extending k by adjoining an indeterminate.

Let $T_1, T_2, \ldots, T_r$ be a basis over k of $\overline{\mathfrak{A}}_i$ for $i \geqslant \min(n, m) + 1$. If T_j is an element of the basis least divisible by X then YT_j and $XT_1, XT_2, \ldots, XT_r$ are linearly independent. These elements are in $\overline{\mathfrak{A}}_{i+1}$ and we have that $\dim(\overline{\mathfrak{A}}_{i+1}) - \dim(\overline{\mathfrak{A}}_i) \geqslant 1$ for $i \geqslant \min(n, m) + 1$.

It remains to be proved that $\dim(\overline{\mathfrak{A}}_{i+1}) - \dim(\overline{\mathfrak{A}}_i) \leqslant 2$ for $i \geqslant \min(n, m) + 1$. Without loss of generality we can assume that $n \leqslant m$ and that the initial forms of f and g are irredundant. Since the field k is infinite we can, if necessary by making suitable linear transformation in x and y, assume that X does not divide the initial form of f. Then we show that there exists a g^* divisible by x and such that f and g^* generate $\mathfrak{A}$ and their initial forms are irredundant. If x divides g then there is nothing to prove hence assume that x does not divide g. By Weierstrass preparation theorem

$$f = \delta\left[y^n + c_1(x)y^{n-1} + \cdots + c_n(x)\right] \quad \text{and}$$

$$g = \eta\left[y^p + d_1(x)y^{p-1} + \cdots + d_p(x)\right]$$

where p is some integer bigger than or equal to m, δ, η are units and $c_1(x), \ldots, c_n(x), d_1(x), \ldots, d_p(x)$ are power series in x divisible by x. Then $g^* = g - \eta\delta^{-1}y^{p-n}f$ is as desired. Let $g^* = xg_1$ and $\mathfrak{V}$ be the ideal generated by f and g_1. In view of the lemma it is enough to prove that $\dim(\overline{\mathfrak{V}}_i) - \dim(\overline{\mathfrak{V}}_{i-1}) \leqslant 2$ for $i \geqslant n + 1$. Since $\mathfrak{V}$ is generated by f and g_1, $\beta(\mathfrak{V}) \leqslant \beta(\mathfrak{A})$ and if $\beta(\mathfrak{V}) = \beta(\mathfrak{A})$ then $\alpha(\mathfrak{V}) < \alpha(\mathfrak{A})$, hence the proof of the theorem follows by induction.

Now in the following corollaries we answer the questions raised by Professor Abhyankar. If f and g are nontangential, i.e., if their initial forms are coprime then $H_i(R)$ depends only on the orders of f and g, and we will denote it by $H_i(n, m)$. It is easily checked that $\Sigma_{i=0}^{\infty} H_i(n, m) = mn$ and $H_i(n, m) = 0$ for all $i \geqslant m + n - 1$.

Corollary 1. *If f and g are tangential then $\Sigma_{i=0}^{m+n-1} H_i(R) > mn$.*

Proof. Let $p > 0$ be the degree of the g.c.d. of the initial forms of f and g. It is easily checked that $H_i(R) = H_i(n, m)$ for $0 \leqslant i \leqslant m + n - p - 1$ and $H_{m+n-p}(R) = H_{m+n-p}(n, m) + 1$. By the above theorem, $H_i(R)$ decreases by at most one at each stage for $i \geqslant \min(n, m)$ whereas $H_i(n, m)$ decreases by one at each stage for $i \geqslant \min(n, m)$ until it becomes zero. It follows that $H_i(R) \geqslant H_i(n, m)$ for all i and

$$\sum_{i=0}^{m+n-1} H_i(R) > \sum_{i=0}^{m+n-1} H_i(n, m) = mn.$$

Corollary 2. *$|H_{i+1}(R) - H_i(R)| \leqslant 1$ for all nonnegative integers.*

Proof. For $0 \leqslant i \leqslant \min(n, m)$ this is checked by direct calculation and for $i \geqslant \min(n, m)$ this follows from the theorem.

We deduce the following result of Bertini cited in [**1**].

COROLLARY 3. *Let α be the intersection multiplicity of f and g and e be the largest integer such that $H_i(R) > 0$ for all $i \leqslant e$ then $e \leqslant \alpha + m + n - mn - 2$.*

PROOF. Since, as observed in the proof of Corollary 1, $H_i(R) \geqslant H_i(n, m)$ for all i and $H_i(n, m) = 0$ for $i \geqslant m + n - 1$ hence $\sum_{i=0}^{e} H_i(R) \geqslant \sum_{i=0}^{m+n-2} H_i(n, m) + (e - m - n + 2)$. But $\sum_{i=0}^{e} H_i(R) = \alpha$, the intersection multiplicity and $\sum_{i=0}^{m+n-2} H_i(n, m) = mn$ hence the result follows.

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