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# ON SPLITTING IN FINITE GROUPS

C. R. B. WRIGHT[1]

ABSTRACT. A splitting criterion due to Šemetkov yields complements to residual normal subgroups in finite solvable groups, as well as splitting conditions for nonsolvable groups.

A group $G$ splits over its normal subgroup $H$ in case $G$ has a subgroup $X$ with $H \cdot X = G$ and $H \cap X = 1$. The purpose of this note is to call attention to two conditions under which a finite group must split over a normal subgroup, and to show how one condition follows from the other.

The first condition, stated in the corollary to Theorems 1 and 2 below, arises in the context of the theory of formations of solvable groups. A nonempty class $\mathfrak{F}$ of groups which is closed under taking homomorphic images is a *formation* in case each group $G$ has a unique smallest subgroup, denoted $G^{\mathfrak{F}}$, with factor group $G/G^{\mathfrak{F}}$ in $\mathfrak{F}$. The characteristic subgroup $G^{\mathfrak{F}}$ is called the $\mathfrak{F}$-*residual* subgroup of $G$. The classes of nilpotent groups, supersolvable groups, and $\pi$-groups, for $\pi$ a fixed set of primes, are familiar examples of formations. Each of them has the additional property of being locally induced; for the definition see, for example p. 176 of **[1]**, §3 of **[3]**, or Definition VI.7.3 of **[5]**.

One of the early applications of the theory of formations was the determination of sufficient conditions for a finite solvable group to split over an $\mathfrak{F}$-residual subgroup. Each of **[1]**, **[4]**, **[6]**, **[7]** and **[9]** contains a theorem to the effect that, for certain locally induced formations $\mathfrak{F}$, if $G$ is solvable and if $G^{\mathfrak{F}}$ is sufficiently abelian, then $G$ splits over $G^{\mathfrak{F}}$.

One might ask what special properties of the subgroups $G^{\mathfrak{F}}$ allow one to prove results of this kind. In **[10]** and **[11]** we introduced a notion of *residual normal subgroup* and showed:

THEOREM 1 (PROPOSITION 1.2 OF **[10]**). *Let $K$ be a normal subgroup of the solvable group $G$, and let $\pi$ be the set of primes dividing $|G/K|$. If $\mathfrak{F}$ is a locally induced formation containing the cyclic group of order $p$ for each $p$ in $\pi$, then $K^{\mathfrak{F}}$ is residual in $G$. (In particular, $G^{\mathfrak{F}}$ is residual in $G$ for every locally induced $\mathfrak{F}$.)*

THEOREM 2 (COROLLARY 2.2 OF **[10]**, THEOREM 2 OF **[11]**). *If $H$ is a residual normal subgroup of the finite solvable group $G$, and if $H$ has abelian Sylow*

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*groups for primes dividing $|G/H|$, then $G$ splits over $H$.*

Theorems 1 and 2 together yield the following sufficient splitting condition.

COROLLARY. *Suppose that $\mathfrak{F}$ is a locally induced formation with support $\pi$, that $G$ is finite and solvable, and that $K$ is a normal subgroup of $G$ such that $G/K$ is a $\pi$-group and the $p$-Sylow groups of $K^{\mathfrak{F}}$ are abelian for all $p$ in $\pi$. Then $G$ splits over $K^{\mathfrak{F}}$.*

Most of the splitting theorems of **[1]**, **[4]**, **[6]**, **[7]** and **[9]** deal with special circumstances in which this corollary applies.

Our aim now is to show that Theorem 2 is itself a consequence of a second, more general criterion, due to Šemetkov, which does not seem widely known:

THEOREM 3 (THEOREM 2.2 OF **[8]**). *A finite group $G$ splits over its normal subgroup $K$ in case for each prime divisor $p$ of $|G/K|$ and $p$-Sylow group $G_p$ of $G$, $G_p \cap K$ is abelian and $G_p$ splits over $G_p \cap K$.*

To show the connection between the two theorems, we observe first that residual normal subgroups can be characterized in purely arithmetic terms. For $p$ a prime, let $\mathbf{H}_p/H = \mathbf{O}_p(G/H)$. Then by Proposition 2.5 of **[10]**, a normal subgroup $H$ is residual in $G$ precisely if $\mathbf{H}_p = \mathbf{O}^p(\mathbf{H}_p)$ for each prime $p$ dividing $|G/H|$. We can thus recognize Theorem 2 as the solvable case of the following:

THEOREM 2′. *Let $H \trianglelefteq G$. For each prime $p$ dividing $|G/H|$ let $\mathbf{H}_p/H = \mathbf{O}_{p'}(G/H)$, and suppose that $\mathbf{H}_p = \mathbf{O}^p(\mathbf{H}_p)$, and that $H$ has abelian $p$-Sylow groups. Then $G$ splits over $H$.*

To get Theorem 2′ from Theorem 3, we use the following result, due to Gaschütz.

THEOREM 4 (SATZ 7 OF **[2]**, SATZ IV.3.8 OF **[5]**). *Let $p$ be a prime, and let $K$ be a normal subgroup of the finite group $G$. If $K = \mathbf{O}^p(K)$, and if $K$ has abelian $p$-Sylow groups, then the $p$-Sylow groups of $G$ split over their intersections with $K$.*

Now suppose the hypotheses of Theorem 2′ hold, and let $p$ divide $|G/H|$. By Theorem 4, the $p$-Sylow groups of $G$ split over their intersections with $\mathbf{H}_p$, i.e., over their intersections with $H$. By Theorem 3, $G$ splits over $H$, as claimed.

As another illustration of how Theorems 3 and 4 work in combination, a proof essentially like the one just given shows the following.

COROLLARY. *If $H \trianglelefteq G$, if $H$ is simple (nonabelian), and if $H$ has abelian Sylow groups for primes dividing $|G/H|$, then $G$ splits over $H$.*

Since Theorem 2′ does not assume solvability of $G$, it is more general than Theorem 2. On the other hand, the versions of Theorem 2 given in **[10]** and

[11] yield explicit descriptions of complements to $H$ in $G$, information which seems difficult to extract from the existential argument given here.

The analysis in [10] and [11] was primarily in terms of chief factors, and used the abelian Sylow group hypothesis only at the end, to guarantee well-behaved chief factors. It seems to be difficult to find splitting criteria as generally applicable as Šemetkov's theorem, but with the abelian Sylow group condition replaced by some other restriction on Sylow groups alone. Besides looking in that direction to improve Theorem 2′, one might reasonably try to extend the chief factor methods to make them more useful in the nonsolvable case.

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Department of Mathematics, University of Oregon, Eugene, Oregon 97403