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ON FINITE SIMPLE GROUPS WITH A SELF-CENTRALIZATION SYSTEM OF TYPE $(2(n))$

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ABSTRACT. Let G denote a simple group with a self-centralization system of type $(2(n))$, where $n > 3$. Let X_1 denote an exceptional character of G, then $X_1(1) = kn + 2\varepsilon$ where $\varepsilon = \pm 1$. It is known that

$$|G| = nX_1(1)(X_1(1) - \varepsilon)(ln + 1)$$

where l is a nonnegative integer. In this paper G is classified if $l = 0$, $\varepsilon = 1$ and $X_1(1)$ is odd.

Let G be a finite simple group, a proper subgroup A of G is called a CC subgroup if $C_G(a) \subseteq A$ for all $a \in A^{\#}$. If $|N_G(A)|/|A| = 2$ and $|A| = n$, then G is said to have a self-centralization system of type $(2(n))$. The classification of simple groups with a self-centralization system of type $(2(n))$ is still incomplete. If $n = 3$, then $G \cong \mathrm{PSL}(2, q)$, $q = 5$ or 7 [2]. If $n > 3$, it is well known [5] that G has $(n - 1)/2$ irreducible characters X_i, and one nonprincipal irreducible character Y such that $X_1(1) = kn + 2\varepsilon$, $Y(1) = kn + \varepsilon$ where $\varepsilon = \pm 1$ and $|G| = nX_1(1)Y(1)(ln + 1)$ where l is a nonnegative integer. In all the known simple groups of type $(2(n))$, $l = 0$ [5]. In this paper we classify all simple groups G with a self-centralization system of type $(2(n))$ where $l = 0$, $\varepsilon = 1$, and $X_1(1)$ is odd. In particular we prove the following:

THEOREM A. *Let G be a finite simple group with a self-centralization system of type $(2(n))$ where $n \geqslant 5$. Let A be a subgroup of order n and let X_1 be an exceptional character of G associated with $N(A)$. Assume $X_1(1) = kn + 2$, $X_1(1)$ is odd and $|G| = n(kn + 2)(kn + 1)$, then G is isomorphic to* $\mathrm{Sz}(q)$ *or* $\mathrm{PSL}(2, 2^r)$.

Let $G \in$ *Hypothesis* A if G satisfies the hypothesis of Theorem A but not the conclusion. Let τ be an involution in $N(A)$, Theorem 5.1 [5] implies G has one class of involutions. If S is a set, let $|S|$ denote the number of elements in S.

Assume $G \in$ Hypothesis A.

If $|C_G(\tau)| = 2^r$, then [6] implies either G satisfies Theorem A, $G \cong \mathrm{PSL}(3, 4)$ or $G \cong \mathrm{PSL}(2, q)$ where q is odd, $n = (q + 1)/2$ or $(q - 1)/2$ where n is odd. However $X_1(1)$ is even for $\mathrm{PSL}(3, 4)$ or $\mathrm{PSL}(2, q)$, q odd. Let G_2 be a Sylow 2 subgroup of G; if G_2 is abelian, then [7] again implies a

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contradiction. Hence, $G \in$ Hypothesis A implies $|C_G(\tau)| \neq |G_2|$ and G_2 is not abelian.

If $z \in C_G(\tau)^\#$, let $F_z = \{\tau^g | \tau^g \in C_G(z)\}$.

LEMMA 1. *Assume $G \in$ Hypothesis* A; *then there is an element $x \in C_G(\tau)^\#$ such that x has odd order and $F_x \neq F_\tau$.*

PROOF. Assume x an element of odd order in $C_G(\tau)^\#$ implies $F_x = F_\tau$. $G \in$ Hypothesis A implies $|F_\tau| \neq 1$. Let $\tau_2 \in F_\tau - \{\tau\}$; then $\tau_2 = \tau^g$ for some $g \in G$. Now $\tau_2 \in F_x$ implies $x \in C_G(\tau_2) = (C_G(\tau))^g$. Hence $x = y^g$ where y has odd order and $y \in C_G(\tau)^\#$. Therefore $F_x = F_y g = (F_y)^g = (F_\tau)^g = F_{\tau_2}$. Hence $F_\tau = F_{\tau_2}$. Let $\langle F_\tau \rangle$ be the group generated by F_τ, then $F_{\tau_2} = F_\tau$ implies $\langle F_\tau \rangle$ is an abelian 2-group. Thus $\Omega_1(G_2)$ is abelian and Goldschmidt [3] implies $G \notin$ Hypothesis A.

PROOF OF THEOREM A. We will assume $G \in$ Hypothesis A and obtain a contradiction. Let Y be the nonprincipal nonexceptional character associated with $N(A)$. Since $|G_2| \mid Y(1)$, $Y(\tau) = 0$ and Lemma 6 of [4] implies $|C_G(\tau)| = Y(1)$. Theorem 17.4 [1] implies $Y(z) = 0$ for $z \in C_G(\tau)^\#$. Since $\varepsilon = 1$, Lemma 4 [4] implies $X_i(z) = 1$ for $z \in C_G(\tau)^\#$, and $i = 1, \ldots, (n-1)/2 = t$.

Let T denote the principal character of $C_G(\tau)$ and let T^* denote the character of G induced by T. Let 1_G denote the principal character of G. Frobenius Reciprocity now implies $(T^*, X_i) = 2$ for $i = 1, \ldots, t$, and $(T^*, Y) = (T^*, 1_G) = 1$. Now $T^*(1) = |G|/Y(1) = nX_1(1)$ and $\Sigma_{i=2}^t 2X_i(1) + Y(1) + 1_G(1) = nX_1(1)$ imply $T^* = \Sigma_{i=1}^t 2X_i + Y + 1_G$. Hence, $z \in C_G(\tau)^\#$ implies $T^*(z) = n$.

If $z \in C_G(\tau)^\#$, then $T^*(z)$ is the number of involutions in $C_G(z)$; hence $|F_z| = n$. Let $x \in C_G(\tau)^\#$ where x has odd order and let $z = x\tau$. Now $F_z = F_x \cap F_\tau$. Hence $|F_z| = n = |F_\tau| = |F_x|$ implies $F_\tau = F_z = F_x$. This contradicts Lemma 1.

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