A KOROVKIN-TYPE THEOREM IN
LOCALLY CONVEX $M$-SPACES

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Abstract. Let $E$ be a locally convex $M$-space, $\emptyset \not= M$ a subset. The universal Korovkin-closure of $M$ as well as the sequentially or stationary defined Korovkin-closures coincide with the space of $M$-harmonic elements and with the uniqueness closure of $M$.

1. The Theorem. Let $E, F$ denote locally convex vector lattices ([6], but not necessarily separated); $L(E, F)_+$ is the cone of continuous positive linear operators from $E$ into $F$ and $V(E, F)$ the set of all continuous linear lattice homomorphisms from $E$ into $F$. We write $E'_+$ for $L(E, R)_+$ and $V(E)$ for $V(E, R)$. For $\emptyset \not= M \subset E$ the universal Korovkin closure $K(M)$ ([9], [10]) is defined by $e \in K(M)$

\[
\begin{align*}
&\text{iff for any locally convex vector lattice } F \\
&\text{and for any equicontinuous net } T_a \in L(E, F)_+ \\
&\text{and for any } S \in V(E, F) \text{ the relation}
\end{align*}
\]

\[
\lim_a T_a f = S f \text{ for all } f \in M \implies \lim_a T_a e = S e.
\]

Let $K_a(M)$ and $K_0(M)$ denote the set of all $e \in E$ that satisfy $\ast$, when in $\ast$ the word “net” is replaced by “sequence” and “stationary sequence”, respectively.

$L(M)$ is the closed linear hull of $M$. Let $\hat{M}$ be the set of all finite infima of elements in $L(M)$, i.e.

\[
\hat{M} = \left\{ \bigwedge A | A \not= \emptyset \subset L(M), \text{ } A \text{ finite} \right\}.
\]

Then the set $H(M)$ of $M$-harmonic elements is defined as $H(M) = \hat{M} \cap -\hat{M}$ ([8], [9]). Note that $-\hat{M} = \hat{M}$ is the set of all finite suprema of elements of $L(M)$.

By $U(M)$ we denote the uniqueness closure of $M$, i.e. $e \in U(M)$ iff for all $\mu \in E'_+, \delta \in V(E)$ equality of $\mu$ and $\delta$ on $M$ implies $\mu(e) = \delta(e)$ ([10]; cf. [2]).

A locally convex vector lattice $E$ is called a locally convex $M$-space, if its topology is generated by a family $\{ \| \cdot \|_a \}$ of lattice seminorms which satisfy $\| e \vee f \|_a = \| e \|_a \vee \| f \|_a$ for all positive $e, f \in E$ (“espaces de Kakutani” in [5], cf. [7, II §7]). Such seminorms will be called $M$-seminorms in the sequel.
Theorem. Let $E$ be a locally convex $M$-space and $\emptyset \neq M$ a subset. Then $H(M) = K(M) = K_0(M) = K_0(M) = U(M)$.

Before proving the theorem let us compare its statement with results obtained by other authors.

When $E = C(X)$, $X$ a compact metric space, and $M$ is a point-separating subset containing a strict positive function, the equality $K_0(M) = U(M)$ was proved by H. Berens and G. G. Lorenz in [2]. Since here $M$ is an arbitrary subset of $E$, we have a new result even in this case.

The equality $H(M) = K(M)$ was proved by M. Wolff in [9] for locally convex vector lattices, if the closed linear hull $L(M)$ of $M$ is nearly positively generated in the sense that $L(M) = L(M)_+ - L(M)_-$. If in addition $E$ is dual atomic (i.e. $E'$ is atomic), then $K(M) = U(M)$ as was proved also by M. Wolff in [10]. Since locally convex $M$-spaces are not dual atomic in general, Wolff's theorem and our theorem cover different cases.

As a class of locally convex lattices where our theorem could be applied, one has the class of so-called weighted-function spaces $C^0(X)$, which have been studied by many authors ([3]). They include well-known spaces as for example $E = C(X)$, $X$ completely regular, equipped with the topology of compact convergence or $E = CB(X)$, the bounded continuous real functions on $X$, with the strict topology.

2. Proof of the Theorem. An essential tool in proving the Theorem is the concept of upper and lower envelopes of elements $e \in E$. This concept in connection with Korovkin-theorems is not new: it has been used already by H. Bauer ([1]), H. Berens and G. G. Lorenz ([2]) and by K. Donner ([4]).

Let $E$ be a vector lattice equipped with a lattice seminorm $|| || (|e| < |f|)$ implies $||e|| < ||f||)$, $E'$ its (topological) dual and $B$ the positive part of the unit ball of $E'$. In the weak topology $\sigma(E', E)$ the set $B$ is compact. The evaluation map $E \rightarrow A_0(B)$ sends elements $e \in E$ in continuous affine functions $\hat{e}$ on $B$ vanishing at $0 \in B$. Therefore we can define upper and lower envelopes of $e \in E$ as

$$\hat{e}(\mu) = \inf \{ \mu(f) + r|f \in \hat{M}, r, f, f + r > \hat{e} \},$$

$$\check{e}(\mu) = \sup \{ \mu(f) + r|f \in \hat{M}, r, f, f + r < \hat{e} \}$$

for all $\mu \in B$ (pointwise order on $B$).

We collect some simple properties of the envelopes in the following lemma:

**Lemma 1.** Let $e \in E, \mu \in B$.

(i) $\hat{e}(\mu) \leq \mu(e) \leq \hat{e}(\mu)$ with equality on $L(M)$.

(ii) $(-e)'(\mu) = -\check{e}(\mu)$.

(iii) $e < 0$ implies $\hat{e}(\mu) < 0$.

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1To use this kind of envelope was suggested by K. Donner at the June 1977 meeting on “Riesz spaces and order bounded linear transformations” in Oberwolfach.
(iv) \( \hat{e}(\mu) \leq \|e\| \).
(v) The map \( e \mapsto \hat{e}(\mu) \) is a sublinear functional on \( E \).

We omit the easy proof.

Now, let \( V(E)_1 = V(E) \cap B \) and denote by \( E(M) \) the set of all \( e \in E \) the upper and lower envelopes of which coincide on \( V(E)_1 \).

**Lemma 2.** Let \( e \in E, \delta \in B \). Then there exists a \( \mu \in B \) such that \( \mu(e) = \hat{e}(\delta) \) and \( \mu = \_M \delta \).

**Proof.** By (v) of Lemma 1 the mapping \( p_\delta : f \mapsto f(\delta) \) is a sublinear functional on \( E \). The linear functional \( \mu_0 \) on \( \mathbb{R} \cdot e \) defined by \( \mu_0(re) = r\hat{e}(\delta) \) is dominated by \( p_\delta \); this is evident for \( r > 0 \). For \( r < 0 \) it follows by \( \mu_0(-e) = -\hat{e}(\delta) \leq -\delta(e) = \delta(-e) = (\delta(e))' = p_\delta(\delta) \) using (i) of Lemma 1.

The Hahn-Banach theorem yields an extension \( \mu \) of \( \mu_0 \) dominated by \( p_\delta \) on \( E \). By (iv) of Lemma 1, \( \mu \) is continuous with norm \( \leq 1 \). By (iii) of Lemma 1 it is positive and thus belongs to \( B \). Finally \( \mu \leq p_\delta \) implies \( \mu = \_M \delta \) as \( \delta(f) = \hat{f}(\delta) \) on \( L(\mathcal{M}) \).

**Lemma 3.** We have \( U(M) = E(M) \).

**Proof.** Suppose \( e \) belongs to \( U(M) \) and let \( \delta \in V(E)_1 \). By Lemma 2 there exists a \( \mu \in B \) such that \( \mu(e) = \hat{e}(\delta) \) and \( \mu = \_M \delta \). As \( e \in U(M) \), \( \mu(e) = \delta(e) \) and \( \delta(e) = \hat{e}(\delta) \). Since also \( -e \in U(M) \), \( \delta(-e) = -\delta(e) = (-e)'(\delta) = -\hat{e}(\delta) \) and \( \delta(e) = \hat{e}(\delta) \). Thus \( e \) belongs to \( E(M) \).

Conversely, let \( e \in E(M) \) and choose \( \mu \in E_+^*, \delta \in V(E) \) such that \( \mu = \_M \delta \). By multiplying by a positive constant, if necessary, we can assume \( \mu \in B, \delta \in V(E)_1 \). Now let \( f \in \hat{M} \) and write \( f = \sqrt{\sum_{i=1}^n f_i} \) with \( f_i \in L(M) \) (\( i = 1, \ldots, n \)). Since \( \delta \) is a lattice homomorphism and since \( \mu = \_M \delta \), it follows that

\[
\delta(f) = \sqrt{\sum_{i=1}^n \delta(f_i)} = \sqrt{\sum_{i=1}^n \mu(f_i)} \leq \mu(f)
\]

and

\[
\delta(f) + r \leq \mu(f) + r \quad \text{for all } r \in \mathbb{R}.
\]

The definition of lower envelopes yields \( \hat{e}(\delta) \leq \hat{e}(\mu) \). Similarly, one obtains \( \hat{e}(\mu) \leq \hat{e}(\delta) \). Thus \( e \in E(M) \) implies—using (i) of Lemma 1—\( \mu(e) = \delta(e) \) and \( e \in U(M) \).

**Lemma 4.** Let \( E \) have an \( M \)-seminorm. If \( e \in E \) satisfies \( \delta(e) = \hat{e}(\delta) \) for all \( \delta \in V(E)_1 \), then \( e \in \hat{M} \).

**Proof.** First observe that \( \hat{e} \leq \hat{f} + r, r \in \mathbb{R} \), implies \( r > 0 \) since \( 0 \in B \). Suppose \( \varepsilon > 0, \delta \in V(E)_1 \), by hypothesis on \( e \) there exist \( f \in \hat{M}, 0 < r \in \mathbb{R} \), such that \( \hat{e} \leq \hat{f} + r \) and

\[
(\frac{1}{2} \delta)(f) + r \leq \hat{e}(\frac{1}{2} \delta) + \frac{1}{2} \varepsilon = \frac{1}{2}(\delta(e) + \varepsilon) \leq \frac{1}{2}(\delta(f) + r + \varepsilon).
\]

Thus \( 0 < r < \varepsilon, \hat{e} \leq \hat{f} + \varepsilon \) and \( \delta(f) < \delta(e) + \varepsilon \). Hence the sets \( U_f = \{ \delta \in \)
Let $V(E)$ be a locally convex $M$-space and $\delta(f) < \delta(e) + \varepsilon$ be a $\sigma(E', E)$-open covering of the $\sigma(E', E)$-compact set $V(E)$, when $f$ varies in $\hat{M}$ such that $\delta(f) < \delta(e) + \varepsilon$ for all $i = 1, \ldots, n$ and $V(E)_i = \bigcup_{i=1}^{n} U_i$.

Let $f = \bigwedge_{i=1}^{n} f_i$; then $f \in \hat{M}$ and for an arbitrary $\delta \in V(E)_i$ we have

$$-\varepsilon + \delta(e) < \delta(f) = \min_{i=1, \ldots, n} \delta(f_i) < \delta(e) + \varepsilon.$$ 

Thus $\sup_{\delta \in V(E)_i} |\delta(f - e)| < \varepsilon$. Since $E$ has an $M$-seminorm, $V(E)_i$ contains the extreme points of $B$, so that $\sup_{\delta \in V(E)_i} |\delta(f - e)| = ||f - e|| < \varepsilon$. As $\varepsilon > 0$ was arbitrary, $e \in \hat{M}$ as required.

Now we are able to prove the Theorem.

**Proof of the Theorem.** The inclusion $H(M) \subset K(M)$ was proved by M. Wolff in [8] for arbitrary locally convex vector lattices. The inclusions $K(M) \subset K_p(M) \subset K_q(M) \subset U(M)$ follow immediately by the definitions of the respective spaces.

To prove $U(M) \subset H(M)$ we proceed as follows. Let $\{||\|_\alpha\|\}_{\alpha \in A}$ be a saturated family of $M$-seminorms generating the topology of $E$ and denote by $E_\alpha$ the space $E$ seminormed by $||\|_\alpha$. Furthermore let $U_\alpha(M)$, $E_\alpha(M)$ and $H_\alpha(M)$ denote the spaces $U(M)$, $E(M)$ and $H(M)$ constructed in $E_\alpha$. By Lemma 3 we have $U_\alpha(M) = E_\alpha(M)$, by Lemma 4 and (ii) of Lemma 1, $E_\alpha(M) \subset H_\alpha(M)$. Now the assertion of the Theorem follows by

$$U(M) \subset \bigcap_{\alpha \in A} U_\alpha(M) = \bigcap_{\alpha \in A} E_\alpha(M) \subset \bigcap_{\alpha \in A} H_\alpha(M) = H(M).$$

In [10] M. Wolff proved $U(M) = K(M, I)$ for the identity Korovkin closure $K(M, I)$ of $M$ in $E$ (i.e. in the definition (•) only $F = E$ and $S = I$, the identity on $E$, is allowed), if $E$ is a locally convex $M$-space. Thus our theorem together with Wolff's result implies:

**Corollary.** If $E$ is a locally convex $M$-space, then the identity Korovkin closure of $M$ in $E$ is universal and coincides both with the set of $M$-harmonic elements and the uniqueness closure of $M$.

**References**


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