LOCAL PROPERTIES OF QUOTIENT ANALYTIC SPACES

KUNIO TAKIJIMA AND TADASHI TOMARU

ABSTRACT. Let \( T := \mathbb{C}/\mathbb{Z} \omega_1 + \mathbb{Z} \omega_2 \) be a complex 1-torus and \( E_n \) the set of all elliptic functions of order \( n \). Then M. Namba showed that \( E_n \) is a 2\( n \)-dimensional complex manifold. Let \( \text{Aut} T \) be the automorphism group of \( T \), then \( \text{Aut} T \) is a 1-dimensional compact complex Lie group and the orbit space \( E_n/\text{Aut} T \) is an analytic space. In this paper, we shall show that \( E_n/\text{Aut} T \) has only rational singularities and if \( n > 5 \), \( E_n/\text{Aut} T \) is rigid.

1. Introduction. Let \( M \) be a complex manifold and \( G \) a properly discontinuous transformation group on \( M \). Then H. Cartan [2] showed that the quotient space \( M/G \) is a normal analytic space and D. Burns [1] has proved that \( M/G \) is rational. Moreover M. Schlessinger [7], [8] showed that if \( \text{codim} \ F(G) > 3 \), \( M/G \) has only rigid singularities, where

\[
F(G) := \bigcup_{g \in G - \{e\}} F(g), F(g) := \{ x \in M; gx = x \}
\]

is the set of all fixed points.

In this paper, let \( M \) be also a complex manifold and \( G \) a complex Lie transformation group whose action is proper on \( M \). Then H. Holmann [4] proved that the quotient space \( M/G \) is a normal analytic space. We shall show the following

**Theorem 1.** \( M/G \) has only rational singularities. Moreover if \( \text{codim} \ S(M/G) > 3 \), \( M/G \) is rigid, where \( S(M/G) \) is the set of all singular points of \( M/G \).

Let \( \omega_1, \omega_2 \in \mathbb{C}; \ \text{Im}(\omega_2/\omega_1) > 0 \), \( T := \mathbb{C}/\mathbb{Z} \omega_1 + \mathbb{Z} \omega_2 \) be a complex 1-torus and \( E_n \) the set of all elliptic functions of order \( n \). Then \( E_n \) is a 2\( n \)-dimensional complex manifold [5] and the automorphism group \( \text{Aut} T \) acts on \( E_n \) naturally. Then we have

**Theorem 2.** \( E_n/\text{Aut} T \) is rational and if \( n > 5 \), \( E_n/\text{Aut} T \) has only rigid singularities.

2. The proof of Theorem 1. We shall give some definitions. Let \((X, \mathcal{O}_X)\) be an analytic space and \( \pi: (\tilde{X}, \mathcal{O}_{\tilde{X}}) \to (X, \mathcal{O}_X) \) a resolution of singularities of \( X \). Then \( x \in X \) is called a rational singularity if \( (R^i \pi_* \mathcal{O}_{\tilde{X}})_x = 0 \) for any \( i > 0 \)

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Let \( R \) be an equivalence relation on \( X \). We identify the graph of \( R \) with \( R \). Then \( R \) is called open (resp. proper, finite) if the natural projection \( p_1: R \to X \) is open (resp. proper, finite). And \( R \) is called analytic if the graph \( R \) is an analytic set in \( X \times X \) (cf. [4], [9]).

Let \( M \) be a complex manifold and \( G \) a complex Lie transformation group on \( M \). Then \( G \) is called proper on \( M \) if the graph mapping \( \Psi: G \times M \to M \times M, \Psi(g, x) := (gx, x) \) is proper. (This is different from the above definition that an equivalence relation \( R \) is proper.) Let \( G(x) := \{ gx; g \in G \} \) be the orbit through \( x \) and \( G_x := \{ g \in G; gx = x \} \) an isotropy group at \( x \).

We shall prove Theorem 1 and its proof is essentially owing to the technique by H. Holmann [3], [4]. Let \( \Phi: G \times M \to M, \Phi(g, x) := gx \) be the natural holomorphic mapping. Then \( G(x) = \Phi(G \times \{ x \}) \) is a complex submanifold of \( M \) and biholomorphic to \( G/G_x \). Since \( G \) is proper on \( M \), \( M/G \) is Hausdorff with respect to the quotient topology. Hence from Hilfssatz 7 of [4], for any \( x \in M \) there exist an open connected neighborhood \( U \subset M \) of \( x \) and a submanifold \( N \subset U \) such that

1. \( x \in N \),
2. \( U = (G(x) \cap U) \times N \),
3. \( G(y) \cap U = (G(x) \cap U) \times N_y \) for any \( y \in N \), where \( N_y := G(y) \cap N \) is finite and \( N_x = \{ x \} \). Let \( R \) be an equivalence relation on \( N \) defined by \( R(y) := N_y \) for any \( y \in N \). Then \( R \) is open proper finite analytic (if we choose \( U \) small enough) and \( M/G \) is locally isomorphic to \( N/R \) (cf. the proof of Satz 15 in [4]).

For any \( g \in G_x \), there exists an open connected neighborhood \( V \subset N \) of \( x \) such that \( g' := qg: V \to g'(V) \subset N \) is biholomorphic, where \( q: U \to N \) is the natural projection and the inverse of \( g' \) is

\[(g')^{-1} := q^{-1} \cdot g^{-1}.\]

Let \( G'(V) \) be the set of all biholomorphic mappings on \( V \) induced from \( G_x \) for any connected neighborhood \( V \subset N \) of \( x \). Since \( G \) is proper on \( M \), there exists an open connected neighborhood \( V \subset N \) of \( x \) such that \( G'(V) \) is a finite transformation group on \( V \) and \( R \) coincides with the equivalence relation induced by \( G'(V) \) on \( V \) (cf. the proof of Satz 19 in [4]). Therefore \( M/G \) is locally isomorphic to \( V/G'(V) \) and so \( M/G \) has only rational singularities. If we show the following lemma, the proof of Theorem 1 is complete.

**Lemma 1.** Let \( \Gamma \) be a finite subgroup of \( \text{GL}(n, \mathbb{C}) \). If \( \operatorname{codim} S(\mathbb{C}^n/\Gamma) > 3 \), \( \mathbb{C}^n/\Gamma \) is rigid.

**Proof.** Let \( \Gamma_0 \) be the normal subgroup of \( \Gamma \) generated by reflections, where \( g \in \text{GL}(n, \mathbb{C}) \) is called a reflection if \( g \) is order-finite and \( \operatorname{codim} F(g) = 1 \).
Then $C^n/\Gamma_0$ is biholomorphic to $C^n$ and $\Gamma/\Gamma_0$ acts on $C^n/\Gamma_0$ naturally. Moreover $C^n/\Gamma \cong (C^n/\Gamma_0)/(\Gamma/\Gamma_0)$ and $\Gamma/\Gamma_0$ has no reflection (cf. [6]). Thus we may assume that $\Gamma$ has no reflection. Then $p(F(\Gamma)) = S(C^n/\Gamma)$, where $p$: $C^n \rightarrow C^n/\Gamma$ is the projection.

In fact, $p(F(\Gamma)) \supseteq S(C^n/\Gamma)$ is clear. Let $x \in C^n$ and $p(x) \notin S(C^n/\Gamma)$, then there exists a nonsingular neighborhood $U \subset C^n/\Gamma$ of $p(x)$ such that $p$: $p^{-1}(U) \rightarrow U$ is a finite covering between complex manifolds. We have

$$F(\Gamma) \cap p^{-1}(U) = \{ t \in p^{-1}(U); Jp(t) = 0 \},$$

where $Jp$ is the Jacobian of $p$. Since codim $F(\Gamma) > 2$, $F(\Gamma) \cap p^{-1}(U)$ must be empty. Thus $x \notin F(\Gamma)$. Therefore codim $F(\Gamma) = \text{codim} S(C^n/\Gamma) \geq 3$ and so $C^n/\Gamma$ is rigid.

Remark. If $G$ is not proper on $M$, locally $M/G$ is not always isomorphic to $V/G'(V)$.

3. The proof of Theorem 2. Since $\text{Aut} T$ is compact, $\text{Aut} T$ is proper on $E^n$. Hence the first part of Theorem 2 is straightforward from Theorem 1. Thus we must calculate codim $S(E^n/\text{Aut} T)$.

Let $S_n$ be the $n$-symmetric group. Then $S_n \times S_n$ acts on $C^{2n+1}$ as follows:

$$(S_n \times S_n) \times C^{2n+1} \rightarrow C^{2n+1},$$

$$(g_1, g_2, \alpha, \beta, \gamma) \mapsto (g_1 \alpha, g_2 \beta, \gamma),$$

where $\alpha = (\alpha_1, \ldots, \alpha_n), \beta = (\beta_1, \ldots, \beta_n)$. We define

$$\phi: C^{2n+1} \rightarrow C^{2n+1},$$

$$\phi(\alpha, \beta, \gamma) := (\sigma_1(\alpha), \ldots, \sigma_n(\alpha), \sigma_1(\beta), \ldots, \sigma_n(\beta), \gamma),$$

where $\sigma_i$ is the $i$-fundamental symmetric function $(i = 1, \ldots, n)$. Then we have $C^{2n+1}/(S_n \times S_n) \cong \phi(C^{2n+1}) = C^{2n+1}$.

For $\omega_1, \omega_2 \in C$; $\text{Im}(\omega_2/\omega_1) > 0$, we put $\Delta := Z\omega_1 + Z\omega_2$, $\Delta^* := \Delta - \{0\}$ and

$$\tilde{S}_\Omega := \left\{ (\alpha, \beta, \gamma) \in C^{2n+1}; \sum_{i=1}^n \sigma_i - \sum_{i=1}^n \beta_i = \Omega, \gamma \neq 0, \alpha_i - \beta_j \notin \Delta, \alpha_i - \alpha_j \notin \Delta^*, \beta_i - \beta_j \notin \Delta^* (i,j = 1, \ldots, n) \right\}$$

for $\Omega \in \Delta$. Since $\tilde{S}_\Omega$ is an $S_n \times S_n$-invariant complex manifold,

$$S_\Omega := \phi(\tilde{S}_\Omega) = \{(x, y, z) \in C^{2n+1}; x_1 - y_1 = \Omega, z \neq 0, \alpha_i - \beta_j \notin \Delta, \alpha_i - \alpha_j \notin \Delta^*, \beta_i - \beta_j \notin \Delta^* (i,j = 1, \ldots, n) \}$$

is also a $2n$-dimensional complex manifold, where $\alpha_i, \beta_j$ are roots of

$$X^n - x_1 X^{n-1} + x_2 X^{n-2} + \cdots + (-1)^n x_n = 0,$$  \hspace{1cm} (1)

$$X^n - y_1 X^{n-1} + y_2 X^{n-2} + \cdots + (-1)^n y_n = 0$$  \hspace{1cm} (2)

respectively. And $\tilde{S} := \bigcup_{\Omega \in \Delta} \tilde{S}_\Omega, S := \bigcup_{\Omega \in \Delta} S_\Omega$ are complex manifolds.
By the classical theory of elliptic functions, for any \( f(w) \in E_n \), let \( \alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_n \) be zeroes and poles of \( f(w) \) respectively, then
\[
\sum_{i=1}^{n} \alpha_i - \sum_{i=1}^{n} \beta_i = \Omega \in \Delta,
\]
\[
\Omega = p_0\omega_1 + q_0\omega_2,
\]
\[
f(w) = \gamma \exp 2(p_0\eta_1 + q_0\eta_2)w \prod_{i=1}^{n} \frac{\sigma(w - \alpha_i)}{\sigma(w - \beta_i)},
\]
where \( \eta_j := \zeta(\frac{1}{2}\omega_2)(j = 1, 2) \) and \( \sigma, \zeta \) are the Weierstrass functions. We define \( \tilde{F}: \tilde{S} \to E_n \)
\[
\tilde{F}(\alpha, \beta, \gamma) := \gamma \exp 2(p_0\eta_1 + q_0\eta_2)w \prod_{i=1}^{n} \frac{\sigma(w - \alpha_i)}{\sigma(w - \beta_i)},
\]
where
\[
\sum_{i=1}^{n} \alpha_i - \sum_{i=1}^{n} \beta_i = p_0\omega_1 + q_0\omega_2 \in \Delta.
\]
Since \( \tilde{F} \) is \( S_n \times S_n \)-invariant, \( \tilde{F}: \tilde{S} \to E_n \) induces \( F: S \to E_n \)
\[
F(x, y, z) := z \exp 2(p_0\eta_1 + q_0\eta_2)w \prod_{i=1}^{n} \frac{\sigma(w - \alpha_i)}{\sigma(w - \beta_i)},
\]
where \( x_1 - y_1 = p_0\omega_1 + q_0\omega_2 \in \Delta \) and \( \alpha_i, \beta_i \) are roots of (1), (2) respectively. Then we can show the following lemma (cf. [5, Remark, p. 75]).

**Lemma 2.** \( F: S \to E_n \) is surjective open holomorphic and locally biholomorphic.

We shall calculate \( \text{codim} S(E_n/\text{Aut } T) \) using this locally biholomorphic mapping \( F \). Here we assume that \( \omega_2/\omega_1 \neq e^{\pi i/2}, e^{2\pi i/3} \) under the modular group. Then \( \text{Aut } T \) has two connected components \( T_0, T_1 \) and \( T_0 \) is isomorphic to \( T \) as Lie group. We put
\[
g_1(x_1, x_2, x_3, n) := -x_3 + \frac{n-2}{n} x_1 x_2 - \frac{(n-1)(n-2)}{3n^2} x_3^3,
\]
\[
g_2(x_1, x_2, x_3, x_4, x_5, n) := -x_5 + \frac{n-4}{n} x_1 x_4 - \frac{(n-3)(n-4)}{n^2} x_1^2 x_3
\]
\[
+ \frac{2(n-2)(n-3)(n-4)}{3n^3} x_1^2 x_2 - \frac{(n-1)(n-2)(n-3)(n-4)}{5n^4} x_1^5,
\]
\[
\rho(x, \lambda) := \lambda^n + x_1\lambda^{n-1} + \cdots + x_{n-1}\lambda + x_n, x = (x_1, \ldots, x_n),
\]
\[
a_i(x, \lambda) := \frac{1}{(n-i)!} \rho^{(n-i)}(x, \lambda)
\]
\[
= \binom{n}{i} \lambda^i + \binom{n-1}{i-1} x_1 \lambda^{i-1} + \cdots + (n-(i-1)) x_{i-1} \lambda + x_i,
\]
\[
a'_i(x, \lambda) := (-1)^i a_i(x, \lambda), \quad t := - (\lambda + 2x_1/n).
Then we have the following lemma by long elementary calculation.

**Lemma 3.**

\[ a'(x, \lambda) = a(x, t), \quad a''(x, \lambda) = a_2(x, t), \]
\[ a''(x, \lambda) = a_3(x, t) + 2g_1(x_1, x_2, x_3, n), \]
\[ a_4'(x, \lambda) = a_4(x, t) + 2(n-3)(t + x_1/n)g_1(x_1, x_2, x_3, n), \]
\[ a_5'(x, \lambda) = a_5(x, t) + (n-3)(n-4)(t^2 + 2x_1t/n)g_1(x_1, x_2, x_3, n) \]
\[ + 2g_2(x_1, x_2, x_3, x_4, x_5, n), \]
\[ a_6'(x, \lambda) = a_6(x, t) + (n-3)(n-4)\left( \frac{n-5}{3}t^3 + \frac{n-5}{n}x_1t^2 - \frac{1}{3n^2}x_1^3 \right)g_1 \]
\[ \cdot (x_1, x_2, x_3, n) + 2(n-5)(t + x_1/n)g_2(x_1, x_2, x_3, x_4, x_5, n). \]

In general,

\[ a_i'(x, \lambda) = a_i(x, t) + \sum_{j=0}^{i-3} \sum_{k=0}^{i-j}(\sum_{k=0}^{i-j}(-1)^k\binom{n-k}{i-j-k}\left( \frac{2}{n} x_1 \right)^{i-j-k} \]
\[ \cdot x_k - \binom{n-i+j}{j} x_i \cdot t^j \]
\[ (i = 1, \ldots, n). \]

Let

\[ A_n := \{(x, y, z) \in S_0; a_i'(x, -2x_1/n) = x_i, \]
\[ a_i'(y, -2y_1/n) = y_i (i = 1, \ldots, n)\}. \]

(If \( i = 1, 2, a_i'(x, -2x_1/n) = x_i \) and \( a_i'(y, -2y_1/n) = y_i \) are always true.) Then \( A_n = \{(x, y, z) \in S_0; g_1(x_1, x_2, x_3, n) = g_1(y_1, y_2, y_3, n) = 0 \} \) for \( n = 3, 4 \) and \( A_n = \{(x, y, z) \in S_0; g_1(x_1, x_2, x_3, n) = g_2(x_1, x_2, x_3, x_4, x_5, n) = g_1(y_1, y_2, y_3, n) = g_2(y_1, y_2, y_3, y_4, y_5, n) = 0 \} \) for \( n = 5, 6. \) Thus if \( n > 5 \), we have \( \text{codim} A_n \geq 4. \)

Now we shall prove the rest of Theorem 2. For any \( f(w) \in E_n \), there exists \( (\alpha, \beta, \gamma) \in S_0 \) such that

\[ f(w) = \gamma \exp 2(p_0 \eta_1 + q_0 \eta_2)w \prod_{i=1}^n \frac{\sigma(w - \alpha_i)}{\sigma(w - \beta_i)}, \]
\[ \Omega = p_0 \omega_1 + q_0 \omega_2. \]

Let \( B_n := \{(x, y, z) \in S; (1) \) and \( (2) \) have no simple root\}, then \( B_n \) is an analytic set in \( S \). If \( (a, b, \gamma) := \phi(\alpha, \beta, \gamma) \in (S - B_n) \cup S_0 \), we may assume that \( (a, b, \gamma) \in S_0 \). Then
\[ f(w) = \gamma \prod_{i=1}^{n} \frac{\sigma(w - \alpha_i)}{\sigma(w - \beta_i)}; \quad \alpha_i = b_i. \]

For any \( a(w) \in T_0; a(w) = w - \lambda \), we have
\[ f \circ a(w) = \gamma \prod_{i=1}^{n} \frac{\sigma(w - \lambda - \alpha_i)}{\sigma(w - \lambda - \beta_i)}. \]

Hence \( (\lambda + \alpha, \lambda + \beta, \gamma) \in \tilde{S}_0 \cap F^{-1}(f \circ a(w)) \) and \( \phi(\lambda + \alpha, \lambda + \beta, \gamma) = (a_1(a, \lambda), \ldots, a_n(a, \lambda), a_1(b, \lambda), \ldots, a_n(b, \lambda), \gamma) \in S_0 \cap F^{-1}(f \circ a(w)). \)

For any \( a(w) \in T_1; a(w) = -w - \lambda \), we have
\[ f \circ a(w) = \gamma \prod_{i=1}^{n} \frac{\sigma(w + \lambda + \alpha_i)}{\sigma(w + \lambda + \beta_i)}. \]

Hence \( (-\lambda - \alpha, -\lambda - \beta, \gamma) \in \tilde{S}_0 \cap F^{-1}(f \circ a(w)) \) and \( \phi(-\lambda - \alpha, -\lambda - \beta, \gamma) = (a_1(a, \lambda), \ldots, a_n(a, \lambda), a_1(b, \lambda), \ldots, a_n(b, \lambda), \gamma) \in S_0 \cap F^{-1}(f \circ a(w)). \) Namely the orbit \( (\text{Aut } T)(f(w)) \) is locally described by
\[ \{(a_1(a, \lambda), \ldots, a_n(a, \lambda), a_1(b, \lambda), \ldots, a_n(b, \lambda), \gamma); \lambda \in T \} \cup \{(a_1'(a, \lambda), \ldots, a_n'(a, \lambda), a_1'(b, \lambda), \ldots, a_n'(b, \lambda), \gamma); \lambda \in T \} \text{ in } S_0. \]

Hence using Lemma 3, we can show that \( (\text{Aut } T)_f(w) = \{w\} \) if \( (a, b, \gamma) \notin A_n. \) Then \( E_n/\text{Aut } T \) is nonsingular at \( p(f(w)) \) by the proof of Theorem 1. If \( n > 5, \) we have \( \text{codim } A_n > 4 \) and \( \text{codim } B_n > n > 5. \) Therefore \( \text{codim } S(E_n/\text{Aut } T) > 3 \) and so \( E_n/\text{Aut } T \) is rigid.

In the case \( \omega_2/\omega_1 \equiv e^{\pi i/2} \text{ or } e^{2\pi i/3}, \) we can also show that if \( n > 5, \) \( \text{codim } S(E_n/\text{Aut } T) > 3 \) by almost the same argument as above.

**Example.** Let \( (w) \) be the Weierstrass \( e \)-function. Then
\[ \varphi(w) = \frac{1}{\sigma(w - \alpha)} \frac{\sigma(w - \alpha)^2 \sigma(w + \alpha)^2}{\sigma(w)^4} \in E_4(\varphi(\alpha) = 0) \]

and \( (\text{Aut } T)_{\varphi(w)^2} = \{w, -w\}. \) Thus \( (\alpha, \alpha, -\alpha, -\alpha, 0, 0, 0, 0, 1/\sigma(\alpha)^4) \in \tilde{S}_0 \cap F^{-1}(\varphi(w)^2) \) and \( (0, -2\alpha^2, 0, \alpha^4, 0, 0, 0, 0, 1/\sigma(\alpha)^4) \in S_0 \cap F^{-1}(\varphi(w)^2). \)

By Lemma 3 and the proof of Theorems 1 and 2, we may take \( N = \{x_1 = y_1 = 0\} \subset S_0 \) and for any \((x, y, z) = (0, x_2, x_3, x_4, 0, y_2, y_3, y_4, z) \in N, \) the orbit \( (\text{Aut } T)(F(x, y, z)) \) is locally described in \( S_0 \) as follows:
\[ \{(4\lambda, 6\lambda^2 + x_2, 4\lambda^3 + 2x_2\lambda + x_3, \lambda^4 + x_2\lambda^2 + x_3\lambda + x_4, 4\lambda, 6\lambda^2 + y_2, \}
\[ 4\lambda^3 + 2y_2\lambda + y_3, \lambda^4 + y_2\lambda^2 + y_3\lambda + y_4, z); \lambda \in T \}
\]
\[ \cup \{-(-4\lambda, 6\lambda^2 + x_2, -4\lambda^3 - 2x_2\lambda - x_3, \lambda^4 + x_2\lambda^2 + x_3\lambda + x_4, -4\lambda, \}
\[ 6\lambda^2 + y_2, -4\lambda^3 - 2y_2\lambda - y_3, \lambda^4 + y_2\lambda^2 + y_3\lambda + y_4, z); \lambda \in T \}. \]

Hence \( N_{(x,y,z)} = \{(0, x_2, x_3, x_4, 0, y_2, y_3, y_4, z), (0, x_2, -x_3, x_4, 0, y_2, -y_3, y_4, z)\}. \) Therefore \( E_4/\text{Aut } T \) is isomorphic to \( \{w^2 - uw = 0\} \times C^5 \subset C^8 \) at \( p \) \( (\varphi(w)^2) \) and so not rigid.
References