THE HYPERINVARIANT SUBSPACE LATTICE OF
A CONTRACTION OF CLASS $C_0$

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Abstract. It is shown that if $T$ is a $C_0$ contraction with finite defect
indices, then Hyperlat $T$ is (lattice) generated by those subspaces which are
either $\ker \psi(T)$ or $\text{ran} \xi(T)$, where $\psi$ and $\xi$ are scalar-valued inner
functions.

For a bounded linear operator $T$ on a complex Hilbert space $H$, Hyper-
lat $T$ denotes the lattice of all hyperinvariant subspaces for $T$, that is, the
lattice of those subspaces which are invariant for all operators commuting
with $T$. Recently, Fillmore, Herrero and Longstaff [1] showed that on a
finite-dimensional space $H$, Hyperlat $T$ is (lattice) generated by those
subspaces which are either $\ker p(T)$ or $\text{ran} q(T)$, where $p$ and $q$ are
polynomials. In this note we generalize this to the following

Theorem. Let $T$ be a contraction of class $C_0$ with finite defect indices acting
on a separable Hilbert space. Then Hyperlat $T$ is (lattice) generated by those
subspaces which are either $\ker \psi(T)$ or $\text{ran} \xi(T)$, where $\psi$ and $\xi$ are scalar-
valued inner functions.

Recall that a contraction $T (\|T\| < 1)$ is of class $C_0$ if $T^*x \to 0$ for all $x$.
The defect indices of $T$ are, by definition, $d_T = \text{rank}(1 - T^*T)^{1/2}$ and $d_{T^*} = \text{rank}(1 - TT^*)^{1/2}$. If $T$ is of class $C_0$, then $d_T < d_{T^*}$. For operators $T, T'$
acting on $H, H'$, respectively, $T \sim T'$ means that there exists a family of
operators $\{X_a\}$ from $H$ to $H'$ such that (i) for each $\alpha, X_a$ is one-to-one, (ii)
$\bigvee_a X_a H = H'$, and (iii) for each $\alpha, X_a T = T'X_a$. If $T \sim T'$ and $T' \sim T$,
then $T, T'$ are said to be completely injection-similar, and this is denoted by
$T \sim T$. For contractions of class $C_0$ and with $d_T = m < \infty, d_{T^*} = n < \infty$,
there has been developed a Jordan model which is, in a certain sense,
alogous to the Jordan model for finite matrices. More specifically, if $T$ is
such a contraction then it is completely injection-similar to a uniquely
determined Jordan operator of the form

$$S(\psi_1) \oplus \cdots \oplus S(\psi_k) \oplus S_{n-m},$$

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where \( \varphi_i \)'s are nonconstant inner functions satisfying \( \varphi_{i-1} | \varphi_i \), \( S(\varphi_i) \) denotes the operator on \( H^2 \otimes \varphi_i \otimes H^2 \) which is the compression of the multiplication by \( z \) to the space \( H^2 \otimes \varphi_i \otimes H^2, i = 1, \ldots, k \), and \( S_{n-m} \) denotes the unilateral shift operator on \( H^2_{n-m} \). For more details, the readers are referred to [3].

We first prove our theorem for the case when \( T \) is a Jordan operator.

**Lemma 1.** Let \( T = S(\varphi_1) \oplus \cdots \oplus S(\varphi_k) \) be a Jordan operator. Then Hyperlat \( T \) is (lattice) generated by those subspaces which are either \( \ker \psi(T) \) or \( \operatorname{ran} \xi(T) \), where \( \psi \) and \( \xi \) are scalar-valued inner functions.

**Proof.** Let \( K \in \text{Hyperlat } T \) and for \( i = 1, \ldots, k \), let \( T_i = S(\varphi_i) \). Uchiyama [4] showed that \( K \) corresponds to a regular factorization

\[
\begin{bmatrix}
\varphi_1 & 0 \\
\vdots & \ddots \\
0 & \varphi_k
\end{bmatrix}
\begin{bmatrix}
\xi_1 & 0 \\
\vdots & \ddots \\
0 & \xi_k
\end{bmatrix}
\begin{bmatrix}
\psi_1 & 0 \\
\vdots & \ddots \\
0 & \psi_k
\end{bmatrix}
\]

of the characteristic function

\[
\begin{bmatrix}
\varphi_1 & 0 \\
\vdots & \ddots \\
0 & \varphi_k
\end{bmatrix}
\]

of \( T \), where \( \xi, \psi \) satisfy \( \xi_{i-1} | \xi_i, \psi_{i-1} | \psi_i, i = 2, \ldots, k \). Also

\[
K = \sum_{i=1}^{k} \bigoplus (\xi_i H^2 \otimes \varphi_i H^2).
\]

We claim that \( K = \bigvee_{i=1}^{k} [\ker \psi_i(T) \cap \operatorname{ran} \xi_i(T)] \).

Since for each \( i \), \( \xi_i H^2 \otimes \varphi_i H^2 = \ker \psi_i(T) = \operatorname{ran} \xi_i(T) \), one inclusion is trivial. To prove the other, fix \( j, 1 < j < k \), and let \( x = \sum_{i=1}^{k} \bigoplus x_i \) be an element in \( \ker \psi_j(T) \cap \operatorname{ran} \xi_j(T) \). Let \( \{ y_n = \sum_{i=1}^{k} \bigoplus y_{in} \} \) be a sequence of vectors such that \( \xi_j(T)y_n \to x \) in norm. Thus for each \( i \), we have \( \xi_j(T_i)y_{in} \to x_i \). For \( i < j \), \( \xi_i | \xi_j \), and therefore there is an inner \( \rho_i \) such that \( \xi_j = \xi_i \rho_i \). Hence \( \xi_j(T_i)\rho_i(T_i)y_{in} = \xi_i(T_i)y_{in} \to x_i \), which implies \( x_i \in \operatorname{ran} \xi_i(T) = \xi_i H^2 \otimes \varphi_i H^2 \).

On the other hand, for \( j < i \), \( \psi_j | \psi_i \), and therefore there is an inner \( \omega_i \) such that \( \psi_j = \omega_i \psi_i \). Hence \( \psi_i(T_j)x_i = \omega_i(T_j)\psi_j(T_j)x_i = 0 \), which implies \( x_i \in \ker \psi_i(T) \).

We remark that in the preceding proof we actually showed that

\[
K = \ker \psi_1(T) \bigvee \left[ \bigvee_{i=2}^{k-1} (\ker \psi_i(T) \cap \operatorname{ran} \xi_i(T)) \right] \bigvee \operatorname{ran} \xi_k(T),
\]

since for \( j = 1, k \), we only used the assumptions \( x \in \ker \psi_1(T) \) and \( x \in \operatorname{ran} \xi_k(T) \) to prove the assertion.

**Lemma 2.** Let \( T = S(\varphi_1) \oplus \cdots \oplus S(\varphi_k) \oplus S_{n-m} \) be a Jordan operator.
Then Hyperlat $T$ is (lattice) generated by those subspaces which are either $\ker \psi(T)$ or $\overline{\text{ran } \xi(T)}$, where $\psi$ and $\xi$ are scalar-valued inner functions.

**Proof.** Let $S = S(\varphi_1) \oplus \cdots \oplus S(\varphi_k)$ and $H = (H^2 \oplus \varphi_1 H^2) \oplus \cdots \oplus (H^2 \oplus \varphi_k H^2)$. Uchiyama showed in [5] that the hyperinvariant subspaces of $T$ must be of the form $K_1 \oplus K_2$, where $K_1 \subseteq H$, $K_2 \subseteq H^2_{n-m}$ are hyperinvariant for $S$, $S^\wedge_{n-m}$, respectively, such that either $K_2 = 0$ or there exists an inner function $\varphi$ such that $K_2 = \varphi H^2_{n-m}$ and $K_1 \supseteq \varphi(S)H$. Note that for any inner function $\varphi$, $\ker \varphi(S^\wedge_{n-m}) = 0$ and $\text{ran } \varphi(S^\wedge_{n-m}) = \varphi H^2_{n-m}$. Thus by Lemma 1 we can easily check that if $K_2 = 0$ then

$$K_1 \oplus K_2 = K_1 \oplus 0 = \bigvee_{i=1}^k \left( \ker \psi_i(T) \cap \overline{\text{ran } \xi_i(T)} \right),$$

otherwise

$$K_1 \oplus K_2 = \overline{\text{ran } \psi(T)} \vee \left[ \bigvee_{i=1}^k \left( \ker \psi_i(T) \cap \overline{\text{ran } \xi_i(T)} \right) \right].$$

This proves our assertion.

**Proof of Theorem.** Let $T$ be completely injection-similar to its Jordan model $T' = S(\varphi_1) \oplus \cdots \oplus S(\varphi_k) \oplus S^\wedge_{n-m}$, and suppose that $T$ and $T'$ are acting on the spaces $H$ and $H'$, respectively. Note that the complete injection-similarity can be implemented by two suitably chosen operators $\{X_1, X_2\}$ from $H$ to $H'$ and two operators $\{Y_1, Y_2\}$ from $H'$ to $H$ (cf. [2] and [3]). Uchiyama [5] showed that in this case the induced mappings $\alpha: K \to X_1 K \vee X_2 K$ and $\beta: K' \to Y_1 K' \vee Y_2 K'$ are (lattice) isomorphisms between Hyperlat $T$ and Hyperlat $T'$, which are inverses to each other. Thus in view of Lemmas 1 and 2 to complete the proof we have only to show that (i) $\beta(\ker \psi(T')) = \ker \psi(T)$ and (ii) $\beta(\overline{\text{ran } \xi(T')}) = \overline{\text{ran } \xi(T)}$ hold for arbitrary $\psi, \xi$ in $H^\infty$.

To prove (i), let $x = Y_1 y$, where $y \in \ker \psi(T')$. Since $\psi(T)x = \psi(T)Y_1 y = Y_1 \psi(T') y = 0$, we have $x \in \ker \psi(T)$. This shows that $Y_1 \ker \psi(T') \subseteq \ker \psi(T)$. Similarly, $Y_2 \ker \psi(T') \subseteq \ker \psi(T)$, and hence $\beta(\ker \psi(T')) \subseteq \ker \psi(T)$. In a similar fashion, we have $\alpha(\ker \psi(T)) \subseteq \ker \psi(T')$. Thus $\ker \psi(T) = \beta(\ker \psi(T')) \subseteq \ker \psi(T')$, which proves (i). (ii) can be proved analogously. This finishes the proof of the Theorem.

**Corollary.** Let $T$ be a linear transformation on a finite-dimensional space $H$. Then Hyperlat $T$ is (lattice) generated by those subspaces which are either $\ker p(T)$ or $\overline{\text{ran } q(T)}$, where $p$ and $q$ are polynomials.

**Proof.** For $0 < \alpha < 1/\|T\|$, $S = \alpha T$ is a strict contraction, hence a contraction of class $C_0$. The Theorem implies that Hyperlat $S = \text{Hyperlat } T$ is (lattice) generated by those subspaces which are either $\ker U$ or $\overline{\text{ran } V}$, where $U, V$ are operators in $\{S\}'' = \{T\}''$, the double commutants of $S$ and $T$. Our assertion follows from the fact that $\{T\}''$ consists of polynomials in $T$. 

**Corollary.** Let $T$ be a linear transformation on a finite-dimensional space $H$. Then Hyperlat $T$ is (lattice) generated by those subspaces which are either $\ker p(T)$ or $\overline{\text{ran } q(T)}$, where $p$ and $q$ are polynomials.
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