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ON L^1-CONVERGENCE OF FOURIER SERIES WITH QUASI-MONOTONE COEFFICIENTS

J. W. GARRETT, C. S. REES AND Č. V. STANOJEVIĆ

ABSTRACT. For the class of Fourier series with quasi-monotone coefficients, it is proved that $\|s_n - \sigma_n\| = o(1)$, $n \to \infty$, if and only if $a_n \lg n = o(1)$, $n \to \infty$. This generalizes a theorem for monotone coefficients and provides a new proof for a result due to Telyakovskii and Fomin.

The problem of L^1-convergence of the Fourier cosine series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$

has been settled for various special classes of coefficients. W. H. Young [1] found that

$$a_n \lg n = o(1), \qquad n \to \infty \tag{Y}$$

is a necessary and sufficient condition for cosine series with convex ($\Delta^2 a_n \geqslant 0$) coefficients, and A. N. Kolmogorov [2] extended that result to the cosine series with quasi-convex ($\sum_{n=1}^{\infty} n|\Delta^2 a_{n-1}| < \infty$) coefficients. G. A. Fomin [3] showed that for cosine series with monotone coefficients (Y) is a sufficient condition and

$$a_{2n}^2 / a_n \lg n = o(1), \qquad n \to \infty \tag{$*$}$$

is a necessary one. It is easy to see that $\Delta a_n \geqslant 0$ and ($*$) imply (Y). Hence, for cosine series with monotone coefficients such that ($*$) holds, the condition (Y) is necessary and sufficient for L^1-convergence. J. W. Garrett and Č. V. Stanojević [4] improved that result by showing that for trigonometric series with monotone coefficients such that

$$a_n \lg n = o(1), \quad n \to \infty, \qquad \|s_n - f\| = o(1), \quad n \to \infty \Leftrightarrow f \in L^1$$

where $\|\cdot\|$ is the L^1-norm and

$$s_n(x) = \frac{a_0}{2} + \sum_{k=1}^{n} (a_k \cos kx + b_k \sin kx).$$

(For sine series an analogous result is due to E. Hille and J. D. Tamarkin [5].) Recently, S. A. Telyakovskii and G. A. Fomin [6] obtained a similar result for Fourier series with quasi-monotone coefficients. A sequence $\{a_n\}$, $a_n \to 0$, $n \to \infty$, is called quasi-monotone if for some $\alpha > 0$, the sequence a_n / n^α is

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monotonically decreasing (for $\alpha = 0$, the sequence is monotone).

The proof given in [6] involves certain results of S. M. Lozinskii [7] concerning summability and interpolation processes. In this paper, we shall show that the Telyakovskii-Fomin theorem follows from a generalization of Theorem 1 in [4]. Our proof depends only on an estimate of $\|s_n - \sigma_n\|$, where σ_n is the Fejér sum.

THEOREM. *Let*

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

be a Fourier series with quasi-monotone coefficients. Then $\|s_n - \sigma_n\| = o(1)$, $n \to \infty$, *if and only if* $(a_n + b_n) \lg n = o(1)$, $n \to \infty$.

PROOF. We shall carry out the proof for the cosine series only, the proof for the sine series being essentially the same. For the "if" part, we need the following lemma.

LEMMA. *Let* $\{a_n\}$ *be a quasi-monotone sequence such that* $a_n \lg n = o(1)$, $n \to \infty$. *Then*

$$\frac{1}{n} \sum_{k=1}^{n} k|\Delta a_k| \lg k = o(1), \quad n \to \infty.$$

PROOF. From

$$\frac{1}{n} \sum_{k=1}^{n} a_k = \frac{1}{n} \sum_{k=1}^{n-1} \Delta(ka_k) \sum_{j=1}^{k} \frac{1}{j} + a_n \sum_{j=1}^{n} \frac{1}{j}$$

we obtain

$$\frac{1}{n} \sum_{k=1}^{n-1} k\Delta a_k \sum_{j=1}^{k} \frac{1}{j} = \frac{1}{n} \sum_{k=1}^{n} a_k - a_n \sum_{j=1}^{n} \frac{1}{j} + \frac{1}{n} \sum_{k=1}^{n-1} a_{k+1} \sum_{j=1}^{k} \frac{1}{j}.$$

The quasi-monotonicity of the $\{a_n\}$ yields

$$|\Delta a_k| \leqslant \Delta a_k + 2\alpha a_k/k$$

for some $\alpha > 0$. Hence,

$$\frac{1}{n} \sum_{k=1}^{n-1} k|\Delta a_k| \sum_{j=1}^{k} \frac{1}{j} \leqslant -a_n \sum_{j=1}^{n} \frac{1}{j} + \frac{1}{n} \sum_{k=1}^{n} a_k + \frac{2\alpha}{n} \sum_{k=1}^{n-1} a_k \sum_{j=1}^{k} \frac{1}{j} + \frac{1}{n} \sum_{k=1}^{n-1} a_{k+1} \sum_{j=1}^{k} \frac{1}{j}.$$

Each term on the right-hand side is $o(1)$ as $n \to \infty$. That completes the proof of the lemma.

Now

$$\|s_n - \sigma_n\| = \frac{1}{n+1}\left\|\sum_{k=1}^{n} ka_k \cos kx\right\|$$

$$\leqslant \frac{1}{n+1}\left\|\sum_{k=1}^{n-1} k\Delta a_k\left[D_k(x) - \frac{1}{2}\right]\right\|$$

$$+ \frac{1}{n+1}\left\|\sum_{k=1}^{n-1} a_{k+1}\left[D_k(x) - \frac{1}{2}\right]\right\| + a_n\left\|D_n(x) - \frac{1}{2}\right\|$$

where $D_n(x)$ is the Dirichlet kernel. Or, since $\|D_n(x) - 1/2\| = O(\lg n)$, for some $B > 0$

$$B\|s_n - \sigma_n\| \leqslant \frac{1}{n+1}\sum_{k=1}^{n-1} k|\Delta a_k|\lg k + \frac{1}{n+1}\sum_{k=1}^{n-1} a_{k+1}\lg k + a_n \lg n.$$

From the lemma it follows that

$$\|s_n - \sigma_n\| = o(1), \qquad n \to \infty.$$

For the "only if" part, notice that

$$\|s_n - \sigma_n\| + \|\sigma_n - f\| \geqslant \|s_n - f\| \geqslant C\sum_{k=1}^{n} \frac{a_{n+k}}{k}$$

where C is a positive constant. Since $f \in L^1$, we have that $\|\sigma_n - f\| = o(1)$, $n \to \infty$. Assume that $\|s_n - \sigma_n\| = o(1)$, $n \to \infty$. Then

$$\sum_{k=1}^{n} \frac{a_{n+k}}{k} = o(1), \qquad n \to \infty.$$

From the fact that the sequence $\{a_n\}$ is quasi-monotone, we have

$$\sum_{k=1}^{n} \frac{a_{n+k}}{k} \geqslant n^\alpha \sum_{k=1}^{n} \frac{1}{k}\frac{a_{n+k}}{(n+k)^\alpha} \geqslant \frac{n^\alpha a_{2n}\lg n}{(2n)^\alpha} = \left(\frac{1}{2}\right)^\alpha a_{2n}\lg n.$$

Finally, $a_n \lg n = o(1)$, $n \to \infty$.

The proof of the Telyakovskii-Fomin theorem follows from the observation that if $f \in L^1$, then $\|s_n - \sigma_n\| = o(1)$, $n \to \infty \Leftrightarrow \|s_n - f\| = o(1)$, $n \to \infty$.

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MATH TECH, INC., P. O. BOX 2392, PRINCETON, NEW JERSEY 08540

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF NEW ORLEANS, NEW ORLEANS, LOUISIANA 70122

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF MISSOURI-ROLLA, ROLLA, MISSOURI 65401