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IMMERSIONS OF SEMIANALYTIC SPACES

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ABSTRACT. It is proved that d: $\mathrm{Imm}(V, M) \to L$ is a weak homotopy equivalence, where $\mathrm{Imm}(V, M)$ denotes the space of smooth immersions of a compact semianalytic space V into a manifold M, L denotes the space of continuous bundle maps, linear and injective on each fibre, from the Zariski tangent bundle of V to the tangent bundle of M, and d is the differential. This generalizes the Haefliger-Poenaru-Hirsch-Smale immersion theory for compact manifolds.

1. Locally. Let V be a compact subset of R^n. $C^\infty(V)$ denotes the Fréchet space of smooth functions on V with the Whitney topology. By a theorem of L. M. Graves (see [1] for references to Graves' work) the epimorphism $C^\infty(R^n) \to C^\infty(V)$ induced by restriction has a continuous right splitting E. For any point p of V let $T_p^*(V) = m_p/I(V) + m_p^2$, where m_p denotes the ideal in $C^\infty(R^n)$ of smooth functions vanishing at p and $I(V)$ denotes the ideal of smooth functions vanishing on V, and let $T_p(V)$, the (Zariski) *tangent space* of V at p, be the dual of $T_p^*(V)$. The *dimension* of V is $\mathrm{Max}_{p \in V}(\dim T_p(V))$ and $T(V) = \bigcup_{p \in V} T_p(V)$, the *tangent bundle* of V, is topologised as a subspace of $T(R^n)$.

$f \in C^\infty(V, M)$ is called a (smooth) *immersion* of V in M if for each $p \in V$ there is a neighborhood U_p such that f^*: $C^\infty(M) \to C^\infty(U_p)$ is onto. By the Malgrange preparation theorem [5] this is equivalent to saying that

$$f^*: T^*_{f(p)}(M) \to T^*_p(V)$$

is onto for each $p \in V$, or dually that df_p: $T_p(V) \to T_{f(p)}(M)$, the *differential* of f at p (the dual of f^*), is injective for each $p \in V$, where $T_q(M)$ ($T_q^*(M)$) denotes the ordinary tangent (respectively cotangent) space to M at q. The space of immersions of V in M will be denoted by $\mathrm{Imm}(V, M)$.

Let $x_1, \ldots, x_n$ be coordinate functions for R^n vanishing at p and let $\{x_i; i \in I\}$ be a subcollection such that their images in $T_p^*(V)$ span this space. Then by the Malgrange preparation theorem the local ring of smooth functions on V at p is the image under the projection map π: $R^n \to R^{|I|}$, $(x_1, \ldots, x_n) \to \{x_i; i \in I\}$ of the local ring at $\pi(p)$ of the smooth functions on $R^{|I|}$; in particular $x_j = f_j(x_i)$, $j \notin I$, $i \in I$, for smooth locally defined functions f_j on R^n at p. The equations $x_j = f_j(x_i)$ define a local submanifold U_p of R^n at p of dimension $|I|$ containing V locally at p. If f is in $\mathrm{Imm}(V, M)$ and f' extends f over some neighborhood of p then f' will be an embedding of

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U_p sufficiently small. f can be extended to a mapping $E(f)$ defined on some neighborhood U of V in R^n as follows. Let $i\colon M \to R^m$ be an embedding of M in some Euclidean space, let $\pi\colon W \to M$ be the projection from some tubular neighbourhood W of M in R^m and let $E'\colon C^\infty(V, R^m) \to C^\infty(R^n, R^m)$ be a continuous right inverse for the restriction mapping given by Graves' theorem. Then let $U = E'(f)^{-1}(W)$ and put $E(f) = \pi \circ E'(f)$.

2. Global. Let V now be a compact semianalytic subset of R^n. $V_i' = \{x \in V;\ \dim T_x(V) \geqslant i\}$, $i = 0, 1, \ldots,$ is closed semianalytic and the filtration $V = V_0' \supset V_1' \supset \cdots$ can be refined (see **[10]**) to a filtration by closed semianalytic sets $V = V_0 \supset V_1 \supset \cdots \supset V_{r+1} = \varnothing$ such that

(1) the collection of associated difference sets $\{W_i = V_i - V_{i+1},\ i = 0, \ldots, r\}$ is a stratification by C^1 submanifolds of R^n satisfying the "axiom of the frontier" (Mather [7]), such that for $x \in W_i$ the tangent space $T_x(V)$ has constant dimension and contains $T_x(W_i)$, the ordinary tangent space to W_i at x;

(2) the $(r + 2)$-tuple $(R^n = V_{-1}, V_0, V_1, \ldots, V_r)$ can be triangulated.

In fact the filtration gives analytic W_i but only C^1 strata are needed in what follows, and (2) follows from **[4]**. Only properties (1) and (2) of V will be used in the rest of the paper. By a simple piecing together argument applied to the U_p of §1 for points p of W_i, for decreasing i, there is a collection $\{N_i;\ i = 0, \ldots, r\}$ of differentiable submanifolds of R^n such that $W_i \subseteq N_i$ and

(1) $N_i \cap N_j \neq 0$ for $i > j$ only if $\overline{W_j} \cap W_i \neq \varnothing$;

(2) N_i is diffeomorphic to the unit disk bundle of $(T(V)|W_i)/T(W_i)$, with W_i corresponding to the zero cross section under the diffeomorphism;

(3) for $x \in N_i \cap N_j$, $i > j$, then $T_x(N_i) \supseteq T_x(N_j)$;

(4) $\overline{N_i} - N_i = \overline{W_i} - W_i$.

Let $M_i = \mathrm{cl}(N_i - \bigcup_{j>i} N_j)$, $i = 0, \ldots, r$ be the associated collection of "closure difference sets". By putting the boundaries of the N_i into general position it can be assumed that in addition

(5) the collection of M_i at any point p of R^n is locally differentiably equivalent to a point in a *half-space flag*, where a half-space flag is the collection of closure difference sets of a collection $X_1, X_2, \ldots, X_s$ of half-subspace of R^n of the form

$$X_j = \left\{x_i = 0,\ 1 \leqslant i \leqslant n_j, \text{ but } i \neq n_1, \ldots, n_j,\ x_{n_j} \geqslant 0\right\},$$
$$j = 1, \ldots, s, \text{ where } 0 < n_1 < n_2 \ldots n_s \leqslant n + 1$$

(if $n_s = n + 1$ the condition $x_{n+1} \geqslant 0$ is vacuous).

Let $N = \bigcup_{i=0}^{r} N_i$ and let $L(T(N), T(M))$ denote the space of continuous fibrewise linear-injective bundle maps from $T(N)$ to $T(M)$ with $\{N_i\}$ satisfying conditions (1) to (5) above.

PROPOSITION. $\mathrm{Imm}(N, M)$ *is weakly homotopy equivalent to* $L(T(N), T(M))$ *when* $\dim N < \dim M$, *induced by the differential.*

PROOF. The handlebody approach to immersions [2] adapts immediately to manifolds with corners (as defined, say, in [6]) and we will assume the theory of immersions of these. Since each M_i is a manifold with corners it is sufficient, following [2], to show that the restrictions

$$\mathrm{Imm}(Z_i, M) \to \mathrm{Imm}(Z_{i+1}, M)$$

and

$$L(T(Z_i), T(M)) \to L(T(Z_{i+1}), T(M)), \quad i = 0, \dots, r-1,$$

where $Z_i = \bigcup_{j \geqslant i} N_j$, both have the covering homotopy property with respect to X-parameter families of maps, where X is any compact set. But an X-parameter homotopy can be extended to a neighbourhood of $Z_{i+1} \cap M_i$ in M_i which can be assumed to be a submanifold with corners of M_i and so the results follow from the corresponding theorems for manifolds with corners.

3. The theorem. Let $\{\tilde{N}_i;\ i = 0, \dots, r\}$ be a fixed collection of submanifolds of R^n as in §2 and let C be the collection of all such collections $c = \{N_i,\ i = 0, \dots, r\}$ with $N_i \subseteq \tilde{N}_i$, $i = 0, \dots, r$. For $c' = \{N_i'\}$, $c'' = \{N_i''\}$ which are both in C put $c' < c''$ if $Z_i'' \subseteq Z_i'$ for all i (in the notation of §2). $[X, Y]$ denotes the set of homotopy classes of maps from X to Y.

For f in $\mathrm{Imm}(V, M)$ then $E(f)$ (in the notation of §1) is an immersion of $N = \bigcup_{i=0}^{r} N_i$ for $c = \{N_i\}$ sufficiently small and so there is an induced map, for any compact X,

$$\alpha\colon [X, \mathrm{Imm}(V, M)] \to \lim_{c} [X, \mathrm{Imm}(N, M)],$$

which is clearly bijective.

$$\delta\colon \lim_{c} [X, \mathrm{Imm}(N, M)] \to \lim_{c} [X, L(T(N), T(M))]$$

induced by the differential is bijective by §2. Let D be the collection of $\{U_i,\ i = 1, \dots, r\}$ such that U_i is a closed relative neighbourhood of V_i mod V_{i+1} in V with $U_i - V_{i+1} \subseteq \tilde{N}_i$, ordered in the same way as C. By the neighbourhood deformation retract property of simplicial complexes the natural restriction

$$\beta\colon \lim_{c} [X, L(T(N), T(M))] \to \lim_{d} [X, L(T(\tilde{N})_d, T(M))] = L(X)$$

is bijective, where $T(\tilde{N})_d = \bigcup_{i=0}^{r} T(\tilde{N}_i)|(U_i - V_{i+1})$. Hence the following result holds, with $\partial = \beta\,\delta\alpha$.

THEOREM A. *When V satisfies conditions* (1) *and* (2) *of* §2 *and* $\dim V < \dim M$ ∂ *is bijective* (*for compact* X).

In order to formulate this result in ordinary homotopy theory we will find $d \in D$ such that the natural map

$$[X, L(T(\tilde{N}))_d, T(M)] \to L(X)$$

is bijective.

4. Reformulation. If Y is a subset of X and Z and $\pi: X \to Z$ is a continuous map with $\pi | Y = 1_Y$ then the mapping cylinder of π relative to Y, $M(X, Y; \pi)$, is the quotient if the disjoint union $(X \times I) \cup Z$ under the equivalence given by $(x, 0) \sim \pi(x)$, $x \in X$, and $(y, t) \sim y$ for $(y, t) \in Y \times I$. We also denote by π: $M(X, Y; \pi) \to Z$ the continuous map given by $\pi(x, t) = \pi(x)$ for $(x, t) \in X \times I$ and $\pi | Z = 1_Z$ and let σ and ρ respectively be the continuous maps from $M(X, Y; \pi) - Z \approx X \times I$ obtained by projection onto X and I respectively. X will be naturally identified with $X \times 0$ in what follows and $\frac{1}{2} M(X, Y; \pi)$ will denote the image in $M(X, Y; \pi)$ of $X \times [0, \frac{1}{2}] \cup Z$.

If A is a subcomplex of B then points of the open simplicial neighbourhood U of $A^{(1)}$ in $B^{(1)}$ (the first deriveds) lie on unique line segments joining points of A to points of bdy (U) thus inducing a projection π: $U \to A$, and this gives a natural identification of U_1, the closed simplicial neighbourhood of $A^{(2)}$ in $B^{(2)}$ with the mapping cylinder of $\pi | \partial U_1$. More generally, for the r-tuple $(K_1, \ldots, K_r)$ of subcomplexes of $K = K_0$ (in the notation of §2) the second derived of K gives a sequence of closed neighbourhoods N_i of K_i mod K_{i+1}, $i = 0, \ldots, r$ and homeomorphisms

$$h_i: M_i = M(\partial N_i, K_{i+1}; \pi_i) \to N_i$$

for suitable π_i: $\partial N_i \to K_i$ with $\pi_i | \partial M_i \times 1 \approx \partial N_i$ equal to $1_{\partial N_i}$, such that (identifying each N_i with M_i under h_i)

(1) $N_i \cap N_j \neq \varnothing$ for $i > j$ only if $\mathrm{cl}(K_j') \cap K_i' \neq \varnothing$, where $K_k' = K_k - K_{k+1}$.

(2) $N_i \cap N_j$, $N_i \cap \partial N_j$, $N_i \cap K_j$, for all i and j, are the submapping cylinders of N_i on $\partial N_i \cap N_j$, $\partial N_i \cap N_j$ and $\partial N_i \cap K_j$ respectively.

(3) $\pi_i \pi_j = \pi_i$ for $i > j$; $\sigma_i \sigma_j = \sigma_j \sigma_i$ for $i > j$ on $(N_i \cap N_j) - K_j$.

(4) $\rho_j \sigma_i = \rho_j$ on $(N_i \cap N_j) - K_i$.

A *shrinking isotopy* of K is an isotopy f_t, $t \in I$, of K which is the composition of continuous isotopies $f_{i,t}$, $i = 0, \ldots, r$ such that $f_{i,t} | (K - N_i) \cup K_i$ is the identity and $f_{i,t}(x, s)$ is of the form $(x, \phi_s(t))$ for a monotone decreasing function ϕ_s, for (x, s) in $\partial N_i \times (0, 1] \approx N_i - K_i$. Using shrinking isotopies it can be assumed that $U_i \subseteq \tilde{N}_i$ where $U_i = \frac{1}{2} N_i$ and then it is a simple exercise in the covering homotopy theorem for bundles to show that $d = \{ U_i;\ i = 0, \ldots, r\}$ satisfies the condition at the end of §3 and so Theorem A implies the following result.

THEOREM B. $\mathrm{Imm}(V, M) \to L(T(\tilde{N})_d, T(M))$, *induced by the differential, is a weak homotopy equivalence under the conditions of Theorem* A.

5. Smale theory. Let L be the space of continuous fibre maps, linear and injective on fibres, from $T(V)$ to $T(M)$ and let $L' = L(T(\tilde{N})_d, T(M))$. Choose homotopies γ_i of id: $N_i \to N_i$, $i = 0, \ldots, r$, such that $\gamma_{i0} = \mathrm{id}$, $\gamma_{it} | \partial N_i = \mathrm{id}$, $\pi_i \gamma_{it} = \pi_i$, $\rho_i \gamma_{it}(x)$ decreases with t for x in N_i, and $\gamma_{i1}^{-1}(K_i) = U_i$.

Extend γ_i to K by the identity outside K_i, let Γ be the (homotopy) composition $\gamma_0 \cdot \gamma_1 \cdot \cdots \cdot \gamma_r$ and let

$$N_i' = N_i - \bigcup_{j>i} U_j, \qquad U_i' = N_i' \cap U_i.$$

By the covering homotopy property (and induction on i) there are continuous families of bundle isomorphisms, "covering" Γ_t, τ_{it}: $T(\tilde{N}_i)|N_i' \to \Gamma_t^*(T(\tilde{N}_i))|N_i'$ such that $\tau_{i0} = \mathrm{id}$ and

$$\begin{array}{ccc} T(\tilde{N}_i)|(N_i' \cap N_j') & \xrightarrow{\tau_{it}} & \Gamma_t^*(T(\tilde{N}_i))|(N_i' \cap N_j') \\ \downarrow & & \downarrow \\ T(\tilde{N}_j)|N_j' & \xrightarrow{\tau_{jt}} & \Gamma_t^*(T(\tilde{N}_j))|N_j' \end{array}$$

commutes for $j > i$, where the vertical maps are induced by the inclusion $T(\tilde{N}_i) \subseteq T(\tilde{N}_j)$ on their common domain. Let r: $L' \to L$ be the natural "restriction"; a homotopy inverse γ for r will be found. For f in L let $\gamma f|(T(\tilde{N}_i)|U_i')$ be given by the composition of

$$\tau_{i1}\colon T(\tilde{N}_i)|U_i' \to \Gamma_1^*(T(\tilde{N}_i)|U_i')$$

with the natural map into $T(\tilde{N}_i)$ followed by $f|(T(\tilde{N}_i)|_{K_i'})$ (note that $T(\tilde{N}_i)$ and $T(V)$ coincide over K_i'). Since Γ is homotopic as a map of $(V_0, V_1, \ldots, V_r)$ into itself to the identity, $r\gamma$ is homotopic to id_L, and using $\{\tau_{it}, t \in I\}$ it follows easily that γr is homotopic to $\mathrm{id}_{L'}$. Then by Theorem B we have the main theorem.

THEOREM. *The differential* d: $\mathrm{Imm}(V, M) \to L$ *is a weak homotopy equivalence.*

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