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BORNOLOGICAL SPACES OF NON-ARCHIMEDEAN VALUED FUNCTIONS WITH THE POINT-OPEN TOPOLOGY

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ABSTRACT. F denotes a nontrivially non-Archimedean valued field with rank one, X an ultraregular space and $C(X, F, p)$ is the vector space $C(X, F)$ of all continuous functions from X into F with the topology p of pointwise convergence. We show that $C(X, F, p)$ is a bornological space if and only if X is a $\mathbf{Z}$-replete space. Also, some results are found concerning the compact-open topology c and we make a comparison with that case as studied by Bachman, Beckenstein, Narici and Warner.

Introduction. The paper is self-contained, but closely related to **[1]** where the reader may find an extensive bibliography. As in that work, an ultraregular space is a Hausdorff one in which the clopen (closed-and-open) sets form a base of open sets. Ultraregular spaces are also widely known as zerodimensional spaces; they coincide with the $\{0, 1\}$-completely regular spaces of **[2]** and with the F-completely regular spaces since F itself is ultraregular.

In **[1]** the E-repletions are obtained as completions of uniform structures and E is assumed to admit a compatible, separated, complete uniform structure. We shall, however, retain the older and more general purely topological view of **[2]** since we shall deal with a space without natural group uniformity. If E is an ultraregular space, then X is E-replete (E-compact in the terminology of **[2]**) iff there is no ultraregular space Y that contains X as a dense subspace and is such that each continuous function f from X into E may be extended to a continuous function f' from Y into E. Equivalently, X has to be homeomorphic to a closed subspace of a product of copies of E.

If $\mathbf{F}$ is a topological vector space over F, then a subset V of $\mathbf{F}$ is F-absolutely convex iff $ax + by \in V$ whenever $x, y \in V$ and $|a|, |b| \leqslant 1$. $\mathbf{F}$ is an F-bornological space iff the only absolutely convex sets that absorb all bounded subsets of $\mathbf{F}$ are the absolutely convex neighborhoods of 0.

The set $|F| = \{|a|: a \in F\}$ will be provided with a topology in which all points are discrete, except for the point 0 whose neighborhoods are the usual ones. Then $|F|$ admits a natural ultrametric, defined by

$$d(x, y) = \begin{cases} 0 & \text{if } x = y, \\ \max\{x, y\} & \text{if } x \neq y. \end{cases}$$

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$|F|$ is complete in this ultrametric and therefore realcompact since its cardinality is nonmeasurable. By Theorem 9 in **[1]** an ultraregular space now is $|F|$-replete iff it is $\mathbf{Z}$-replete. Remark also that in **[1]** the notation $|F|$ is used with an entirely different meaning.

We now state the key result of this paper:

THEOREM 1. *$C(X, F, p)$ is an F-bornological space if and only if X is a $\mathbf{Z}$-replete space.*

PROOF OF THE "ONLY IF" PART OF THEOREM 1. Assume that there is a point x_∞ in $\upsilon_{|F|}X \setminus X$ where $\upsilon_{|F|}X$ denotes the $|F|$-repletion of X. If $f \in C(X, F)$, then $|f| \in C(X, |F|)$ so that $|f|$ may be extended to a continuous function $|f|^e \in C(\upsilon_{|F|}X, |F|)$. We set $B = \{f \in C(X, F)\colon\ |f|^e(x_\infty) \leqslant 1\}$ and show that B is an absolutely convex set in $C(X, F)$, that it absorbs all bounded sets in $C(X, F, p)$ and that it is not a neighborhood.

If $f, g \in B$ and $a, b \in F$ with $|a| \leqslant 1$, $|b| \leqslant 1$, then clearly $|af + bg|^e \leqslant \max(|f|^e, |g|^e)$ in all points of X. Hence also $|af + bg|^e(x_\infty) \leqslant \max(|f|^e(x_\infty), |g|^e(x_\infty)) \leqslant 1$, so that $af + bg \in B$.

Now let V be bounded in $C(X, F, p)$. Choose $\lambda \in F$ such that $|\lambda| > 1$. If B does not absorb V, then for all $n \geqslant 1$ there exists an $f_n \in V$ with $|f_n|^e(x_\infty) > |\lambda|^n$. We set

$$U_n = \left\{x \in \upsilon_{|F|}X\colon |f_n|^e(x) \leqslant |\lambda|^{n-1}\right\}.$$

If $x \in X$, there exists $\lambda' \in F$ with $V \subseteq \lambda'\{f \in C(X, F)\colon\ |f(x)| \leqslant 1\}$. Hence for all n we have $|f_n(x)| \leqslant |\lambda'|$; so $|f_n(x)| \leqslant |\lambda|^{n-1}$ and $x \in U_n$ for n sufficiently large. We conclude that $x_\infty \notin \bigcup_{n=1}^\infty U_n \supseteq X$. Now let $(V_n)_{n=1}^\infty$ be obtained via $V_p = \{x \in U_p\colon x \notin U_i \text{ if } i < p\}$. Then $(V_n)_{n=1}^\infty$ is a disjoint countable family of clopen subsets of $\upsilon_{|F|}X$ and covers X. If $f\colon X \to |F|$ is obtained by putting $f(x) = |\lambda|^n$ whenever $x \in V_n \cap X$, then $f \in C(X, |F|)$ but cannot be extended to a continuous function on the whole of $\upsilon_{|F|}X$.

If B is a neighborhood, consider a finite $K \subseteq X$ and $\varepsilon > 0$ so that $B \supseteq \{f \in C(X, F)\colon\ |f(x)| \leqslant \varepsilon \text{ for all } x \in K\}$. In particular $|f|^e(x_\infty) \leqslant 1$ whenever $f(x) = 0$ for all $x \in K$; this is obviously false.

Introduction to the proof of the "if" part of Theorem 1. Let X be $|F|$-replete and consider an F-absolutely convex subset S of $C(X, F)$ that absorbs all p-bounded sets; we intend to prove that S is a p-neighborhood. First some notation is introduced. If $f\colon X \to |F|$ is an arbitrary function, then the set $B(f) = \{g \in C(X, F)\colon |g| \leqslant f\}$ is p-bounded so that there is a $\lambda \in F$ with $B(f) \subseteq \lambda S$. If $f, g \in C(X, |F|)$ and $\lambda, \lambda' \in F$, then the following are obvious:

(1) If $f \leqslant g$, then $B(f) \subseteq B(g)$.

(2) If $|\lambda| \leqslant |\lambda'|$ and $B(f) \subseteq \lambda S$, then $B(f) \subseteq \lambda' S$.

(3) If $B(f) \subseteq \lambda S$, then $B(|\lambda'| f) \subseteq \lambda'\lambda S$.

If either $f \in F^X$ or $f \in |F|^X$ and if K is a subset of X, then $f|_K$ will denote a function equal to f on K and zero outside of K; in particular $1|_K$ is the characteristic function of K.

Put $\mathcal{Q} = \{K \subseteq X\colon K$ is clopen and there is a $\lambda \in F \setminus \{0\}$ with $B(1|_K) \nsubseteq \lambda S\}$. Clearly $\phi \notin \mathcal{Q}$.

LEMMA 1. *If $S \neq C(X, F)$, then $X \in \mathcal{Q}$.*

PROOF. Suppose $B(1) \subseteq \lambda S$ for all $\lambda \neq 0$. Let $f \in C(X, |F|)$ be arbitrary and set $T_n = \{x \in X\colon n \leqslant f(x) < n + 1\}$ for all $n = 0, 1, \ldots$. Suppose $B(f^2) \subseteq \lambda_0 S$; we may assume $|\lambda_0| > 1$. For all $n \geqslant 1$ we set $W_n = \bigcup_{n' < n} T_{n'}$, so that

$$B(f) = B(f|_{W_n}) + B(f|_{(X \setminus W_n)}).$$

Let $\lambda \in F \setminus \{0\}$ be arbitrary. There is an $n \geqslant 1$ with $f^2 \geqslant |\lambda| f$ on $X \setminus W_n$; then we have

$$B(f) \subseteq B(f|_{W_n}) + B(|\lambda|^{-1} f^2) \subseteq \lambda^{-1} S + \lambda^{-1}\lambda_0 S \subseteq \lambda^{-1}\lambda_0 S.$$

Hence $B(f) \subseteq S$ for all $f \in C(X, |F|)$ so that $S \supseteq C(X, F)$.

LEMMA 2. *If $A \in \mathcal{Q}$ and if A is the countable union of the clopen sets A_i $(i = 1, 2, \ldots)$ then there is an i with $A_i \in \mathcal{Q}$.*

PROOF. The sets A_i may be assumed to be disjoint. If the result is not true, then $B(1|_{A_i}) \subseteq \lambda S$ for all i and for $\lambda \neq 0$. Choose λ_0 with $|\lambda_0| > 1$ and define $f\colon X \to |F|$ by

$$\begin{aligned} f(x) &= |\lambda_0|^n \quad \text{if } x \in A_n, \\ &= 0 \quad\ \ \text{if } x \notin A. \end{aligned}$$

If $B(f) \subseteq \lambda_1 S$, then for all n we have

$$\begin{aligned} B(1|_A) &= B(1|_{A_1}) + \cdots + B(1|_{A_n}) + B(1| \textstyle\bigcup_{i>n} A_i) \\ &\subseteq \lambda S + \cdots + \lambda S + \lambda_0^{-n}\lambda_1 S. \end{aligned}$$

Hence $B(1|_A) \subseteq \lambda S$ for all $\lambda \neq 0$, a contradiction.

LEMMA 3. *There exists no infinite set of disjoint members of $\mathcal{Q}$.*

PROOF. Suppose that $\mathcal{Q}$ contains the disjoint members A_n for $n = 1, 2, \ldots$; let $B(1|_{A_n}) \nsubseteq \lambda_n S$ with $\lambda_n \neq 0$. Choose $\lambda \in F$ with $|\lambda| > 1$ and define f from X into $|F|$ by setting

$$\begin{aligned} f(x) &= |\lambda|^n |\lambda_n|^{-1} \quad \text{if } x \in A_n, \\ &= 0 \quad \text{if } x \notin \textstyle\bigcup_{n=1}^{\infty} A_n. \end{aligned}$$

If $B(f) \subseteq \lambda_0 S$, then for all n

$$B(1|_{A_n}) \subseteq \lambda_0 \lambda^{-n} \lambda_n S.$$

If n is chosen so that $|\lambda_0 \lambda^{-n}| \leqslant 1$, a contradiction arises.

DEFINITION. A set $A \in \mathcal{Q}$ is called special iff it is not a disjoint union of two members of $\mathcal{Q}$.

LEMMA 4. *If $S \neq C(X, F)$, then X is a finite union of disjoint special sets $A_1, A_2, A_3, \ldots, A_n$.*

PROOF. By Lemma 1 we have $X \in \mathfrak{C}$. Suppose there is no decomposition of X as described above. Then X is not special, so that there is a partition $\mathfrak{T}_1 = \{T_1^1, T_2^1\}$ of X with T_2^1, T_2^1 in $\mathfrak{C}$. Since at least one of these two sets is not special, there is a partition $\mathfrak{T}_2$ of X, $\mathfrak{T}_2 = \{T_1^2, \ldots, T_{n(2)}^2\}$, that is subordinate to $\mathfrak{T}_1$, with $n(2) > 2$ and all T_i^2 belonging to $\mathfrak{C}$ if $1 \leqslant i \leqslant n(2)$.

By induction one constructs a family $(\mathfrak{T}_p)_{p=1}^{\infty}$ of partitions of X in members of $\mathfrak{C}$ with $\mathfrak{T}_{n+1}$ subordinate to $\mathfrak{T}_n$ for all n and with $\mathfrak{T}_n$ containing more than n elements. A contradiction with Lemma 3 is now easily obtained.

LEMMA 5. *Let $S \neq C(X, F)$ and let $A_1, \ldots, A_n$ be as in Lemma 4. Fix i with $1 \leqslant i \leqslant n$. If $f \in C(X, |F|)$, then one of the following assertions holds*:

(5a) $f^{-1}([0, \lambda[) \cap A_i \in \mathfrak{C}$ *for all* $\lambda \in |F| \setminus \{0\}$,

(5b) *there is a unique* $\lambda \in |F| \setminus \{0\}$ *with* $f^{-1}(\lambda) \cap A_i \in \mathfrak{C}$.

PROOF. Assume that (5a) does not hold; there is a $\lambda_1 \in |F| \setminus \{0\}$ with $f^{-1}([0, \lambda_1[) \cap A_i \notin \mathfrak{C}$. A straightforward application of Lemma 2 shows that there is a $\lambda_2 \in |F|$, $\lambda_2 > \lambda_1$, with $f^{-1}([\lambda_1, \lambda_2[) \cap A_i \in \mathfrak{C}$. Since A_i is special, the case where F has a discrete valuation now becomes easy; therefore we consider a dense valuation.

By successive application of Lemma 2 we construct a sequence

$$\left(\left[\lambda_j^0, \lambda_j^b\right[\right)_{j=1}^{\infty} \quad \text{with } \lambda_1^0 = \lambda_1, \lambda_1^b = \lambda_2, |\lambda_j^0 - \lambda_j^b| \leqslant (2/3)^{j-1}|\lambda_2 - \lambda_1|,$$

$$f^{-1}\left(\left[\lambda_j^0, \lambda_j^b\right[\right) \cap A_i \in \mathfrak{C}, \quad \left[\lambda_j^0, \lambda_j^b\right[\subseteq \left[\lambda_{j'}^0, \lambda_{j'}^b\right[\quad \text{if } j \geqslant j'.$$

Now $A_i \cap \bigcap_{j=1}^{\infty} f^{-1}([\lambda_j^0, \lambda_j^b[) \neq \varnothing$, for otherwise A_i is the countable union of the clopen sets $A_i \setminus f^{-1}([\lambda_j^0, \lambda_j^b[)$ neither of which belongs to $\mathfrak{C}$; this contradicts Lemma 2. Hence there is a unique $\lambda \in |F| \cap \bigcap_{j=1}^{\infty}([\lambda_j^0, \lambda_j^b[)$. Now put $Q_1 = [\lambda_1^0, \lambda_1^b[\setminus \{\lambda\}$ and $Q_j = [\lambda_{j-1}^0, \lambda_{j-1}^b[$ for $j = 2, 3, \ldots$. Then $A_i = \bigcup_{j=1}^{\infty}(A_i \setminus f^{-1}(Q_j))$ so that by Lemma 2 there is a j with $(A_i \setminus f^{-1}(Q_j)) \in \mathfrak{C}$. This is possible only for $j = 1$ so that $A_i \cap f^{-1}([\lambda_1, \lambda_2[\setminus \{\lambda\}) \notin \mathfrak{C}$; hence $A_i \cap f^{-1}(\lambda) \in \mathfrak{C}$.

LEMMA 6. *Let $S \neq C(X, F)$ and let $A_1, \ldots, A_n$ be as in Lemma 4; fix i with $1 \leqslant i \leqslant n$. There is an a_i in A_i with the property that $G \in \mathfrak{C}$ whenever G is a clopen neighborhood of a_i.*

PROOF. Suppose not. Put $X' = X \cup \{x_\infty\}$ with $x_\infty \notin X$. For an arbitrary f in $C(X, |F|)$ define f' from X' into $|F|$ by

$$\begin{aligned} f'(x_\infty) &= 0 \quad \text{if } f^{-1}\left(\left[0, \lambda\right[\right) \cap A_i \in \mathfrak{C} \text{ for all } \lambda \neq 0, \\ &= \lambda \quad \text{if } f^{-1}(\lambda) \cap A_i \in \mathfrak{C}, \\ f'(x) &= f(x) \quad \text{if } x \in X. \end{aligned}$$

(This definition is made possible by Lemma 5.)

If X' is provided with the weak topology induced by all such functions f', then it is easily seen to be ultraregular. It contains X as a dense subspace, which is in contradiction with the assumption that X is an $|F|$-replete space.

Proof of the "if" part of Theorem 1. Let $S \neq C(X, F)$ (otherwise the result is trivial) and consider $A_1, \ldots, A_n$ as in Lemma 4 with $a_1, \ldots, a_n$ as in Lemma 6. Choose $\lambda \in |F| \setminus \{0\}$ such that $B(1) \subseteq \lambda S$ and put $U = \{f \in C(X, F)\colon |f(a_i)| < |\lambda|^{-1}$ for $i = 1, \ldots, n\}$. To show that $U \subseteq S$, consider $f \in U$. For each i there is a clopen G_i with $a_i \in G_i \subseteq A_i$ and $|f| < |\lambda|^{-1}$ on G_i.

Then

$$|f| \leqslant |\lambda|^{-1}(1|_{G_1}) + \cdots + |\lambda|^{-1}(1|_{G_n}) + |f| \cdot (1|_{X \setminus \bigcup_{i=1}^{n} G_i}).$$

An argument similar to the one used in Lemma 1 shows that $B(|f| \cdot (1|_{X \setminus \bigcup_{i=1}^{n} G_i}))$ is a subset of S (in fact of $\lambda' S$ for all $\lambda' \in F \setminus \{0\}$); the key is that $X \setminus \bigcup_{i=1}^{n} G_i \notin \mathcal{Q}$. We then have

$$B(|f|) \subseteq \lambda^{-1}\lambda S + \cdots + \lambda^{-1}\lambda S + S \subseteq S.$$

In particular f belongs to S; this completes the proof of Theorem 1.

Remarks. (1) From [**1**, Theorem 9 and Theorem 13] we infer that the notions of F-repleteness and $\mathbf{Z}$-repleteness are identical if F has a nonmeasurable cardinal. So in all practically occurring cases $C(X, F, p)$ is bornological if and only if X is F-replete.

(2) The first part of the proof of Theorem 1 applies as well to the compact-open topology. Hence if $C(X, F, c)$ is bornological, then X is $\mathbf{Z}$-replete.

(3) If F is complete and indiscrete and has a nonmeasurable cardinal, then by [**1**, Theorem 21] $C(X, F, c)$ is bornological iff $C(X, F, p)$ is bornological. We conjecture that this result is true for all F. The "only if" part follows from Remark 2.

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References

1. G. Bachman, E. Beckenstein, L. Narici and S. Warner, *Rings of continuous functions with values in a topological field*, Trans. Amer. Math. Soc. **204** (1975), 91–112.

2. S. Mrówka, *Further results on E-compact spaces*.I, Acta Math. **120** (1968), 161–185.

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