ON THE COVERING DIMENSION OF SUBSPACES OF PRODUCT OF SORGENFREY LINES

ALI A. FORA

Abstract. Let $S$ denote the Sorgenfrey line. Then the following results are proved in this paper:

(i) If $X$ is a nonempty subspace of $S^\infty$, then $\dim X = 0$.

(ii) For any nonempty separable space $X \subset S^\infty$, $\dim X^m = 0$ for any cardinal $m$.

1. Introduction. The question of whether $\dim(X \times Y) < \dim X + \dim Y$ for topological spaces $X$ and $Y$ has long been considered (see e.g., [G, pp. 263 and 277]). By $\dim X$, or the covering dimension of $X$, we mean the least integer, $n$, such that each finite cozero cover of $X$ has a finite cozero refinement of order $n$. (A cover is of order $n$ if and only if each point of the space is contained in at most $n + 1$ elements of the cover. All spaces considered are completely regular.)

Researchers have long worked on the above problem, and only recently Wage [W] and Przymusiński [P] constructed a Lindelöf space $X$ such that $\dim X = 0$ and $X^2$ is normal but has $\dim X^2 > 0$.

The aim of this paper is to prove that no product of subspaces of Sorgenfrey lines can serve as a counterexample to the product conjecture. Another aim is to give a full answer to one of the questions raised by Mrowka [Mr2] in the conference of 1972 which says: "We still do not know if subspaces of $S^n$ ($n = 2, 3, \ldots, \omega_0$) are strongly 0-dimensional."

The familiar Sorgenfrey space $S$ is defined to be the space of real numbers with the class of all half-open intervals $[a, b)$, $a < b$, as a base. It is a well-known fact that $S$ is Lindelöf, first countable, $N$-compact and also has $\dim S = 0$.

A Tychonoff space $X$ is called strongly zero-dimensional provided that $\dim X = 0$.

The following theorem (see e.g., [G]) characterizes the class of all strongly zero-dimensional spaces.

1.1 Theorem. For a Tychonoff space $X$, the following conditions are equivalent:

Received by the editors August 9, 1976 and, in revised form, October 19, 1977.
AMS (MOS) subject classifications (1970). Primary 54F45; Secondary 54F50.
Key words and phrases. Completely regular, cozero cover, cozero set, $N$-compact, order of a cover, Tychonoff.

The results in this paper are contained in the author's doctoral dissertation, written at SUNY at Buffalo under the direction of Stanislaw Mrowka in February 1976.

© American Mathematical Society 1978
(a) $X$ is strongly zero-dimensional.
(b) $\beta X$ is strongly zero-dimensional.
(c) Every cozero-set of $X$ is a countable union of clopen sets of $X.$

It can be easily seen now that a Lindelöf space which has a base consisting of clopen sets must be strongly zero-dimensional. Since $S^2$ fails to be Lindelöf, there is no easy way to determine $\dim S^2.$ The fact that $\dim S^n = 0$ for all $n$ was proved only in 1972 [Mr1], [Te1]. Prior to that, several researchers have proved that $\dim S^2 = 0$ [Nyikos, Fund. Math. 79 (1973), 131–139], but their arguments could not be generalized, even to $S^3.$ An interesting parallel is that Terasawa (private communication) has shown that $S^2$ is hereditarily strongly zero-dimensional; his proof cannot be generalized even to $S^3.$

2. The covering dimension of subspaces of product of Sorgenfrey lines. It is the time now to discuss the main result of this paper.

2.1 Proposition. Let $Y$ be any strongly zero-dimensional metrizable space and $X \neq \emptyset$ be a subspace of $S^n \times Y$ (where $n$ is a fixed integer). Then $X$ is strongly zero-dimensional.

Proof. Let $X$ be a nonempty subspace of $S^n \times Y.$ By Theorem 1.1, it is sufficient to prove that each cozero-set of $X$ is the union of countably many clopen sets in $X.$

The proof will be carried out in several steps (I–II).

I. For any $z = (z_1, \ldots, z_n) \in S^n,$ define

$$V_i(z) = \left[z_1, z_1 + 1/i\right) \times \cdots \times \left[z_n, z_n + 1/i\right).$$

For each integer $i > 1,$ choose the sequence $(u_{ik})$ in $S^n$ such that $S^n = \bigcup_{k=1}^{\infty} V_i(u_{ik}).$ For simplicity, we let $V(i, k) = V_i(u_{ik}),$ and, for $z = (z_1, \ldots, z_i),$ $A \subset S^n,$ $B \subset Y,$ we let $A \cap x B = (A \times B) \cap X,$ $z(k) = z_k (k < r).$

Since $\dim Y = 0,$ therefore $Y$ has a basis $B = \bigcup_{i=1}^{\infty} B_i$ consisting of clopen sets, where each $B_i$ is a locally finite family (see [E, p. 291]).

II. Let $U$ be any cozero set of $X$ determined by a continuous function $f: X \to [0, 1]$ in such a way that $U = \hat{f}^{-1}(0, 1].$ For natural numbers $i, k, l,$ and for each $G \in B_i,$ define

$$W_G(i, k, l) = \left\{ x = (z, y) \in X \mid f(V_i(z) \cap X G(y)) \subset (1/l, 1], \right\}$$

where $z = (z_1, \ldots, z_n) \in S^n,$ $y \in Y$ and $G(y) = G$

$$\cap V(i, k) \cap X G;$$

$$O_G(i, k, l) = \bigcup \left\{ V_i(z) \cap X G(y) \mid (z, y) \in W_G(i, k, l) \right\}$$

$$\cap V(i, k) \cap X G.$$

(3) If $(z_1, y_1) \in O_G(i, k, l)$ and $(z_2, y_2) \in V(i, k) \cap X G$ such that $z_2(j) >
z(j) (j = 1, \ldots, n), then (z2, y2) ∈ O_G(i, k, l).

Define

\[ \hat{O}_G(i, k, l) = \bigcap_x O_G(i, k, l) \setminus O_G(i, k, l). \]

One can notice that \( \hat{O}_G(i, k, l) \subset V(i, k) \cap \times G \), and that the closure of \( O_G(i, k, l) \) is the same as its Euclidean closure in \( V(i, k) \cap \times G \).

One can also notice that

(5) if \((z_1, y_1) \in \hat{O}_G(i, k, l) \) and \((z_2, y_2) \in V(i, k) \cap \times G \) such that \( z_2(j) > z_1(j) \) (j = 1, \ldots, n), then \((z_2, y_2) \in \bigcap_x O_G(i, k, l) \).

Define:

(6) \( T_G(i, k, l) = \bigcup \{ N(x) : x \in \hat{O}_G(i, k, l) \) and \( f(N(x)) \subset (1/(1 + l), 1] \),

where \( N(x) \) is a basic neighborhood at the point \( x \) \( \cap \bigcap V(i, k) \cap \times G \);

\[ F_G(i, k, l) = \bigcap_x O_G(i, k, l) \cup T_G(i, k, l) \subset V(i, k) \cap \times G. \] (7)

Then \( F_G(i, k, l) \) is a clopen subset of \( X \) (see Observation 1 below).

Define

\[ F(i, k, l) = \bigcup \{ F_G(i, k, l) \mid G \in \mathcal{B}_i \}. \]

Then \( F(i, k, l) \) is clopen in \( X \) because \( \mathcal{B}_i \) is locally finite.

We can easily prove that \( U = \bigcup \{ F(i, k, l) \mid i, k, l \} \) (see Observation 2).

Observation 1. \( F_G(i, k, l) \) is a clopen subset of \( X \).

It suffices to prove that \( F_G(i, k, l) \) is closed since it is clearly open. To show that \( F_G(i, k, l) \) is closed, it suffices to prove that \( \bigcap_x T_G(i, k, l) \setminus T_G(i, k, l) \subset F_G(i, k, l) \). Let \((z, y) \in \bigcap_x T_G \setminus T_G \). If \((z, y) \notin \bigcap_x O_G(i, k, l) \), there exists a basic neighborhood \( N_1 \) at \((z, y)\) such that \((z, y) \in N_1 \subset V(i, k) \cap \times G \) and \( N_1 \cap O_G(i, k, l) = \emptyset \). Since \((z, y) \in \bigcap_x T_G \setminus T_G\), there exists a point \((z_1, y_1) \) \( \in N_1 \cap T_G \). From (6), we can find \((z_2, y_2) \in O_G(i, k, l) \) such that \((z_1, y_1) \in N((z_2, y_2))\). Using (5), we get \((z_1, y_1) \in \bigcap_x O_G(i, k, l) \). Therefore \( N_1 \cap O_G(i, k, l) \neq \emptyset \), which is a contradiction. Therefore \((z, y) \in \bigcap_x O_G(i, k, l) \) and consequently \((z, y) \in F_G(i, k, l) \).

Observation 2. \( U = \bigcup \{ F(i, k, l) \mid i, k, l \} \).

Let \((z, y) \in U \cap S^n \times Y \). By continuity of \( f \), there exist \( l_0, k_0 > 1 \) and \( G \in \mathcal{B}_i \) such that \( f(V_{i_0}(z) \cap G(y)) \subset (1/l_0, 1] \), where \( G(y) = G \). Let \( k_0 > 1 \) be such that \( z \in V_{i_0, k_0} \). Then \((z, y) \in F_G(i_0, k_0, l_0) \) which completes the proposition. We can proceed now to our main theorem.

2.2 Theorem. Let \( Y \) be any strongly zero-dimensional metrizable space and \( X \neq \emptyset \) be a subspace of \( S^{\times n} \times Y \). Then \( X \) is strongly zero-dimensional.

Proof. Let \( U \) be any cozero set of \( X \) determined by a continuous function \( f : X \to [0, 1] \) in such a way that \( U = f^{-1}(0, 1] \). Then \( U = \bigcup_{j=1}^\infty U_j \), where \( U_j = \{ x = (x_1, x_2, \ldots, y) \in U : f(N_1 \times \cdots \times N_j \times S^{\times n} \times G(y) \cap X) \subset (0, 1] \) \), where \( N_1 \times \cdots \times N_j \times S^{\times n} \times G(y) \cap X \) is some basic neighborhood at the point \( x \). For each natural number \( j > 1 \), write \( U_j \) as a countable union of clopen sets (use the same construction as in
Proposition 2.1). It is clear that \( U = \bigcup_{j=1}^{\infty} U_j \) can be written as a countable union of clopen sets which completes the proof of the theorem.

We shall list various corollaries to the above theorem.

2.3 Corollary. The space \( S^m \times Y \) is hereditarily strongly zero-dimensional for every strongly zero-dimensional metrizable \( Y \) and all \( 1 \leq m \leq \kappa_0 \).

2.4 Corollary. If \( X \neq \emptyset \) is a separable subspace of \( S^{\kappa_0} \times Y \), then \( X^m \) is strongly zero-dimensional for all cardinals \( m \).

Proof. Let \( \Lambda: N \to N \times N \) be a fixed bijection. Define the map \( \sim: (S^{\kappa_0} \times Y)^{\kappa_0} \to S^{\kappa_0} \times Y^{\kappa_0} \) by the rule \( \sim (x_1, x_2, \ldots ) = (x_{\Lambda 1}, x_{\Lambda 2}, \ldots , y_{\Lambda 1}, y_{\Lambda 2}, \ldots ) \), where \( x_i = (x_{1, i}, x_{2, i}, \ldots , y_{1, i}) \in S^{\kappa_0} \times Y \) for \( i = 1, 2, \ldots \). It is clear that \( \sim \) is a homeomorphism from \( (S^{\kappa_0} \times Y)^{\kappa_0} \) onto \( S^{\kappa_0} \times Y^{\kappa_0} \).

Let \( \sim (X^{\kappa_0}) = \tilde{X} \). Then \( \tilde{X} \subset S^{\kappa_0} \times Y^{\kappa_0} \) is strongly zero-dimensional and hence \( X^{\kappa_0} \) is also strongly zero-dimensional.

Now, let \( m \) be any cardinal \( \geq \kappa_0 \), and let \( U \) be any cozero set of \( X^m \) which is determined by a continuous map \( f: X^m \to [0, 1] \) in such a way that \( U = f^{-1}(0, 1] \). By the Gleason Theorem (see [I]), we get the existence of continuous maps \( g \) and \( \pi \) such that the following diagram commutes.

\[
\begin{array}{ccc}
X^m & \xrightarrow{f} & [0, 1] \\
\downarrow{\pi} & & \downarrow{g} \\
X^{\kappa_0} & \xrightarrow{\ } & \\
\end{array}
\]

It is clear that \( U = \pi^{-1}(g^{-1}(0, 1]) \) is a countable union of clopen sets in \( X^m \).

2.5 Corollary. If \( \emptyset \neq X \subset S \), then \( X^m \) is strongly zero-dimensional for any cardinal \( m \).

The proof follows immediately from Corollary 2.4 and the fact that \( S \) is a hereditarily separable space.

3. Significance of the main results. As we explained, our results were prompted by the product conjecture "The product of any two strongly zero-dimensional spaces is still strongly zero-dimensional." However, these results are also relevant for other problems, e.g. for the problem of hereditary strong zero-dimensionality of various product spaces, strong zero-dimensionality of \( N \)-compact spaces and also for the following:

The Union Problem \((U)\). If \( X_1 \) and \( X_2 \) are disjoint strongly zero-dimensional subspaces of \( X \), with \( X_1 \) closed in \( X \), and \( X = X_1 \cup X_2 \), then is \( X \) also strongly zero-dimensional?
We wish to discuss our results in view of the above problems. It has been recently demonstrated that $N$-compact spaces need not be strongly zero-dimensional. The first example was given in 1972 by Mrowka [Mr2], and another one, still unpublished, by E. Pol and R. Pol. Both of these examples are quite complex; it is therefore reasonable to inquire whether some well-known space can serve as an example.

As we mentioned before, the product conjecture has been solved negatively by Wage [W] and Przymusiński [P]. The Union Problem also has been solved negatively by Terasawa [Te2].

It is easy to see that $S^{\omega_0}$ is hereditarily $N$-compact. Consequently, for some time, it was conjectured that subspaces of $S^{\omega_0}$ could provide an example of $N$-compact nonstrongly zero-dimensional space. Our results eliminate this possibility; more exactly, they eliminate the following spaces from these considerations:

(a) subspaces of $S^{\omega_0}$,
(b) products of subspaces of $S$.

Observe also that, for large $n$, $S^n$ is not even closed hereditary strongly zero-dimensional. Indeed, if we take the nonstrongly zero-dimensional $N$-compact space $\mu$ (see [Mr2]), then, for large $n$, we have $\mu \subset_{cl} N^n \subset_{cl} S^n$.

Moreover, our results may be considered as generalizations to those found in [Mrl] and [Te1].

To conclude this section we will comment on the connection of our results with the so-called intermediate topology (described in [Ta]). These matters are related to both the Product and the Union Problems. In this matter, one considers a space $X$ with a distinguished subset $M$ such that $M$ and $X \setminus M$ are both metrizable strongly zero-dimensional spaces; the theorem in [Ta] asserts that, under a certain additional assumption, $X$ is strongly zero-dimensional.

On the other hand, by our results, $M \times S^{\omega_0}$ and $(X \setminus M) \times S^{\omega_0}$ are both strongly zero-dimensional. Thus, if $X \times S^{\omega_0}$ (with an $X$ satisfying the assumption of the required theorem in [Ta]) would fail to be strongly zero-dimensional, this would provide a counterexample to the Product Problem as well as to the Union Problem.

**References**


Department of Mathematics, Yarmouk University, Irbid, Jordan