INDECOMPOSABLE DECOMPOSITIONS AND THE MINIMAL DIRECT SUMMAND CONTAINING THE NILPOTENTS

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Abstract. It is well known that an indecomposable right ideal decomposition of a ring is not necessarily unique. In this paper we show that the reduced right ideals of such a decomposition are unique up to isomorphism and the remainder of the decomposition forms the unique MDSN. In the main theorem we use triangular matrices to prove that a ring with an indecomposable decomposition is basically composed of a nilpotent ring, a ring (containing a unity) with an indecomposable decomposition which equals its MDSN, and a direct sum of indecomposable reduced rings with unity.

Generally, nilpotent elements detract from the arithmetic structure of a ring. Various methods (e.g. several types of radicals) have been developed to isolate the nilpotency of a ring. In this paper, we continue the study of the (right) MDSN, the unique minimal direct summand (i.e. idempotent generated right ideal) containing the nilpotent elements of the ring, which was begun in [1] and [2].

Throughout this paper, all rings will be associative; R denotes a ring with unity; N(X) is the set of nilpotent elements of X. As background from [2], we note the following: (1) the MDSN is a semicompletely prime two-sided ideal (i.e. if x^a ∈ MDSN then x ∈ MDSN) which equals the intersection of all direct summands containing the set of nilpotent elements of the ring; (2) if R has a MDSN then so does R[x]; (3) not every ring has a MDSN [2, p. 715], however for rings without a MDSN there is an essential direct sum which approximates the decomposition obtained for rings with a MDSN; (4) in a reduced ring (i.e. without nonzero nilpotent elements) every idempotent is central, and a direct sum of reduced rings is reduced; (5) the complement of the MDSN is a reduced direct summand which is unique up to isomorphism.

Lemma 1. Let e ∈ R such that e is a unity on eR. Then: (i) (1 - e)R and (1 - e)Re are two-sided ideals of R; (ii) (1 - e)R(1 - e) = R(1 - e); (iii) (1 - e)R = (1 - e)Re ⊕ R(1 - e) (left ideal direct sum) hence R = eR ⊕ (1 - e)Re ⊕ R(1 - e) (additive group direct sum); (iv) R is ring isomorphic to

\[
\begin{pmatrix}
eR & 0 \\
(1 - e)Re & R(1 - e)
\end{pmatrix};
\]

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(v) if \( x \in R \) such that \( x^2 = x \), then \( xR = A \oplus B \) where \( B \subseteq (1 - e)R \) and \( A \cap (1 - e)R = 0 \).

**Proof.** The proof is routine as is indicated by the following proof of part (v): \( x = xex + x(1 - e)x \) and \( (xex)^2 = xexex = xex \) since \( e \) is a unity on \( eR \). \( [x(1 - e)x]^2 = x[(1 - e)x(1 - e)]x = x(1 - e)x \) since \((1 - e)x(1 - e) = x(1 - e)\) because \((1 - e)R \) is a two-sided ideal. Also \( (xex)x(1 - e)x = 0 \) since \( e \) is a unity on \( eR \). Thus \( xR = xexR \oplus x(1 - e)xR = A \oplus B \) where \( A = xexR \) and \( B = x(1 - e)xR \). Since \( (xex)(1 - e) = 0 \), \( A \cap (1 - e)R = 0 \).

**Corollary 2.** Let \( R \) be a ring with a MDSN and \( x = x^2 \neq 0 \). Then \( xR = A \oplus B \) where \( A \) is a reduced right ideal and \( B \subseteq \text{MDSN} \).

Thus the MDSN decomposition for a ring is inherited by the direct summands of the ring.

**Lemma 3.** Let \( e \in R \) such that \( e^2 = e \) and \( eR \cap N(R) = 0 \). Then \( N(R) = \{ k \in R | k = x + y \text{ where } x \in (1 - e)Re \text{ and } y \in N(R(1 - e)) \} \).

**Proof.** Straightforward and can be found in [1, Corollary 1.21].

In [4] Osofsky constructs a family of rings in which a ring may have several distinct indecomposable decompositions. The next proposition shows that the nonunique components of an indecomposable decomposition are in the MDSN.

**Proposition 4.** Let \( R = \bigoplus_{i=1}^{\mu} X_i = \bigoplus_{j=1}^{\nu} Y_j \) be decompositions of \( R \) into indecomposable right ideals where \( A = \bigoplus_{i \in I} X_i \) and \( B = \bigoplus_{j \in J} Y_j \) are the direct sums of those \( X_i \) and \( Y_j \) which are reduced with \( X \) and \( Y \) equal to the direct sums of the remaining \( X_i \) and \( Y_j \), respectively. Then \( X = Y = \text{MDSN} \) of \( R \) and there exists a bijection \( f: I \rightarrow J \) such that \( X_i \) is both ring and \( R \)-module isomorphic to \( Y_{f(i)} \) for all \( i \in I \).

**Proof.** From [2, Theorem 2.2, Lemma 1.1, and p. 709], \( X = Y = \text{MDSN} \) of \( R \) and there is a \( R \)-isomorphism \( g: A \rightarrow B \) which is also a ring isomorphism. Let \( T = g(X_i) \) for some \( i \in I \). Then \( T = tR \) where \( t^2 = t \). Since \( t \) is central in \( B \) and the \( Y_j \) are indecomposable right ideals, then there is a \( j \in J \) such that \( T = Y_j \). Consequently, there exists a bijection \( f: I \rightarrow J \) such that \( X_i \) is both ring and \( R \)-module isomorphic to \( Y_{f(i)} \) for all \( i \in I \).

**Theorem 5.** Let \( R = \bigoplus_{i=1}^{\mu} Y_i \) where the \( Y_i \) are indecomposable right ideals, \( R \) is not reduced, and \( R \neq \text{MDSN} \). Then there exists a positive integer \( n < m \) such that for each \( k = 1, \ldots, n \) there is an idempotent \( b_k \) where \( Rb_k = b_k Rb_k \), \( S_k \) is a \((ring) \) direct sum of indecomposable reduced rings with unity, \( X_k \) is a left ideal of \( R \) such that \( X_k^2 = 0 \), and \( Rb_{k-1} \) is ring isomorphic to

\[
\begin{vmatrix}
S_k & 0 \\
X_k & Rb_k
\end{vmatrix}
\]

with \( b_0 = 1 \).

Consequently, \( R = S \oplus X \oplus Rb_n \) (additive group direct sum) where \( S = \).
\( \bigoplus_{k=1}^{n} S_k \) is a (ring) direct sum of indecomposable reduced rings with unity, 
\( X = \bigoplus_{k=1}^{n} X_k \) is a nilpotent left ideal of \( R \), and \( Rb_n \) is a ring (containing a unity) with indecomposable decomposition where \( Rb_n \) equals its MDSN or \( Rb_n \) is reduced.

**Proof.** There exists a complete set of orthogonal primitive idempotents \( \{y_i\}_{i=1}^{n} \) such that \( y_i R = Y_i \). Let \( e = \sum_{i \in I} y_i \) where \( y_i R \) is reduced for \( i \in I \) and \( (1 - e) = \sum_{i \in J} y_i \) where \( y_i R \) is not reduced for \( i \in J \). By hypothesis, \( e \neq 0 \) and \( e \neq 1 \). \( eR \) is a reduced ring (direct sum of indecomposable rings) with unity and \( (1 - e)R \) is the MDSN of \( R \) [2, Theorem 2.2]. Let \( b_1 = 1 - e \), \( X_1 = b_1 R e \), and \( S_1 = e R \). By Lemma 1, \( R = S_1 \oplus X_1 \oplus Rb_1 \) (additive group direct sum), \( X_1 \) is a left ideal of \( R \) such that \( X_1^2 = 0 \), and \( Rb_1 = b_1 R b_1 \) is a ring with unity. If \( Rb_1 \) is reduced or if \( Rb_1 \) equals its MDSN, then we are finished; otherwise, we will continue the decomposition with \( R_1 = Rb_1 \). Thus \( R_1 = \bigoplus_{i \in I} y_i R_1 = e_1 R_1 \oplus (b_1 - e_1) R_1 \) where \( e_1 = \sum_{i \in I} y_i \neq 0 \), and \( b_1 - e_1 = \sum_{i \in J} y_i \) where \( I_1 \cup J_1 = J \); \( e_1 R_1 \) is a reduced ring and \( (b_1 - e_1) R_1 \) is the MDSN of \( R_1 \). Let \( b_2 = b_1 - e_1 \), \( S_2 = e_1 R_1 \), and \( X_2 = b_2 R e_1 = b_2 R e_1 \). Note \( R_1 b_2 = R b_2 \). By Lemma 1, \( R = S_1 \oplus X_1 \oplus S_2 \oplus X_2 \oplus R b_2 \) (additive group direct sum) where \( S_1 \oplus S_2 \) is a ring direct sum of indecomposable reduced rings with unity and \( X_1 \oplus X_2 \) is a nilpotent left ideal of \( R \). If \( R b_2 \) is reduced or if \( R b_2 \) equals its MDSN, then we are finished; otherwise, we will continue with \( R_2 = R b_2 \). Since \( J \) is a finite set this procedure terminates after, say, \( n \) steps. Thus it follows that \( R \) has the desired group direct sum decomposition. The triangular matrix characterization for \( R b_{k-1} \) follows from Lemma 1.

From the above proof and Proposition 4, it follows that the decomposition is independent of the choice of a complete set of orthogonal primitive idempotents in the sense that \( S \) is unique up to ring isomorphism, \( X \) is unique, and \( Rb_n \) is unique. Furthermore, by using left-right symmetry in the above Theorem and [2, Lemma 1.1], we could continue the decomposition with \( R b_n \) (if it is not reduced) on the left to obtain an additive group direct sum of a reduced ring, a nilpotent ring, and a ring which equals its MDSN on the right and on the left.

**Corollary 6.** Let \( R = S \oplus X \oplus Rb_n \) as in Theorem 5. If \( Rb_n \) is a reduced ring then \( X = N(R) \) is a two-sided nilpotent ideal of \( R \).

**Proof.** By Theorem 5 and repeated use of Lemma 3, \( N(R) = X \). Since \( X \) is a left ideal, it follows that \( N(R) \) is a two-sided ideal.

The ring of \( 2 \times 2 \) lower triangular matrices over a field provides an example for Theorem 5 and Corollary 6. Also, this same ring shows that the left MDSN of a ring does not necessarily equal the right MDSN of the ring.

**References**


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