THE CONORMAL MODULE OF AN ALMOST COMPLETE INTERSECTION

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Abstract. The conormal module of an ideal $I$ in a commutative ring $S$ is the $S/I$-module $I/I^2$. Assume $S$ is a regular noetherian ring and $I$ a prime ideal, which is locally everywhere a complete intersection or an almost complete intersection (i.e. needs one generator more than in the complete intersection case). In this situation necessary and sufficient conditions for $I/I^2$ being torsion free are given. Moreover the torsion of $I/I^2$ is expressed in terms of Kahler differentials of $S/I$.

1. Torsion freeness of the conormal module. Let $S$ be a regular local ring, $I$ an ideal of $S$ and $R = S/I$. We say that $I$ (or $R$) is a “complete intersection”, if $\mu(I) = \text{ht}(I)$, and that $I$ (or $R$) is an “almost complete intersection”, if $\mu(I) = \text{ht}(I) + 1$. Here $\mu$ denotes the minimal number of generators and $\text{ht}$ means “height”.

$I$ is a complete intersection iff the conormal module $I/I^2$ is a free $R$-module (see [3] or [9]). In this note we are interested in necessary and sufficient conditions for $I/I^2$ being torsion free, in case $I$ is a prime ideal and an almost complete intersection. Observe that for a prime ideal $I$ the $R$-module $I/I^2$ is torsion free iff $I^2$ is an $I$-primary ideal.

Theorem 1. Let $S$ be a regular noetherian ring, $I$ a prime ideal of $S$ which is locally everywhere a complete intersection or an almost complete intersection. For $R = S/I$ let $K_R$ be the canonical (dualizing) module of $R$, i.e. $K_R = \text{Ext}_S^r(R, S)$, where $r = \text{ht}(I)$. Then the following conditions are equivalent:

(a) $I/I^2$ is a torsion free $R$-module.
(b) $K_R$ is a reflexive $R$-module.
(c) For all $P \in \text{Spec}(R)$ with $\text{ht}(P) = 1$ the local ring $R_P$ is a complete intersection.

From this we see, for example, that if under the assumptions of the theorem we have $\text{dim } R = 1$ and $R_M$ is an almost complete intersection for some maximal ideal $M$ of $R$, then $I^2$ is not primary. Explicit examples in the polynomial ring $K[X_1, X_2, X_3]$ over a field $K$ can easily be given. In fact, it was shown recently that for $I \in \text{Spec}(K[X_1, X_2, X_3])$ the ideal $I^2$ is primary if

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Under the assumptions of the theorem condition (a) is "independent of the embedding", since condition (c) depends obviously only on \( R \).

**Proof of Theorem 1.** It is enough to prove the local version of the theorem, so we shall assume that \( S \) is a regular local ring. We may also assume that \( R \) is an almost complete intersection, since the theorem is known for complete intersections. Matsuoka [7] has constructed an exact sequence

\[
0 \rightarrow K_R \rightarrow R^{r+1} \rightarrow I/I^2 \rightarrow 0.
\]

Moreover, Aoyama ([1, Lemma]) has shown the formula

\[
\text{depth}(K_{R_P}) = \min\{2 + \text{depth}(R_P), \dim R_P\}
\]

for all \( P \in \text{Spec}(R) \).

Let \( C \) be the cokernel of \((I/I^2)^* \rightarrow (R^{r+1})^*\), where * denotes the \( R \)-dual module. Then there is a linear map \( \tau: K_R \rightarrow C^* \) such that the diagram with exact rows

\[
\begin{array}{cccc}
0 & \rightarrow & K_R & \rightarrow & R^{r+1} & \rightarrow & I/I^2 & \rightarrow & 0 \\
& & \downarrow & \tau & & \downarrow & & \downarrow & \\
0 & \rightarrow & C^* & \rightarrow & (R^{r+1})^** & \rightarrow & (I/I^2)^** & \rightarrow & 0
\end{array}
\]

is commutative.

Suppose (a) is satisfied. Then \( I/I^2 \rightarrow (I/I^2)^{**} \) is injective, since \( R \) is a domain, and therefore \( \tau \) is an isomorphism. Since \( C^* \) is reflexive, being the dual of a finitely generated module over a noetherian domain, \( K_R \) is also reflexive. If \( K_R \) is reflexive, then so is \( K_{R_P} \) for all \( P \in \text{Spec}(R) \) with \( \text{ht}(P) = 1 \). By [4, 7.29] \( R_P \) has to be Gorenstein. But \( R_P \) is an almost complete intersection or a complete intersection. By [5] only the second possibility can hold, hence (c) follows from (b).

Assume now that condition (c) of the theorem is satisfied. Then \( \dim R_P > 2 \), if \( R_P \) is an almost complete intersection; hence \( \text{depth}(K_{R_P}) > 2 \) by (2) and \( \text{depth}(R_P \otimes_R I/I^2) > 1 \) by (1). Thus \( P \) is not an associated prime of \( I/I^2 \). If \( R_P \) is a complete intersection, then \( R_P \otimes_R I/I^2 \) is even free. We conclude that \( I/I^2 \) is torsion free.

2. An exact sequence for the torsion of the conormal module. The torsion \( T(I/I^2) \) of \( I/I^2 \) is related to the Kähler and Dedekind different of \( R \) over a suitable subring. In order to simplify we make the following assumptions: \( S = k[X_1, \ldots, X_n] \) is a power series algebra over a perfect field \( k \) and \( I \in \text{Spec}(S) \).

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In $R = S/I$ we write $x_i$ for the image of $X_i$. If $\dim R = d$, there is a power series algebra $Q$ of $d$ variables over $k$, such that $Q \subset R$, $R$ is a $Q$-module of finite type and the quotient field $L$ of $R$ is separable algebraic over the quotient field $K$ of $Q$.

After a change of variables, if necessary, we may assume that $Q = k[x_1, \ldots, x_d]$. We may identify $Q$ with the subalgebra $k[X_1, \ldots, X_d]$ of $S$. Moreover we have an exact sequence

$$0 \to T(I/I^2) \to I/I^2 \to R \otimes_S D_Q(S) \to D_Q(R) \to 0,$$  

(3)

where $D_Q$ is the Kähler differential module relative to $Q$. Suppose now $I$ is an almost complete intersection of height $r = n - d$ and $(F_1, \ldots, F_{r+1})$ a system of generators of $I$. We may assume that the mapping $\beta: R^{r+1} \to I/I^2$ in (1) sends the canonical basis element $e_i$ of $R^{r+1}$ to the image $\overline{F}_i$ of $F_i$ in $I/I^2$ ($i = 1, \ldots, r + 1$). Combining (1) and (3) we get a commutative diagram with exact rows and columns

$$
\begin{array}{ccccccccc}
0 & 0 & & & & & & & \\
\downarrow & \downarrow & & & & & & & \\
K_R & = & K_R & & & & & & \\
\downarrow & \downarrow & & & & & & & \\
0 & \to & D & \to & R^{r+1} & \to & R^r & \to & D_Q(R) & \to & 0 \\
\downarrow & \downarrow & \alpha & \downarrow & \beta & \downarrow & \| & & & & & \\
0 & \to & T(I/I^2) & \to & I/I^2 & \to & R \otimes_S D_Q(S) & \to & D_Q(R) & \to & 0 \\
\downarrow & \downarrow & & & & & & & \\
0 & 0 & & & & & & & 
\end{array}
$$

(4)

where $\alpha$ is given by the Jacobian matrix $J = (\partial F_i/\partial x_k)_{i=1, \ldots, r+1; k=d+1, \ldots, n}$ and $D = \ker(\alpha)$.

**Lemma 1.** $D = \mathcal{D}(R/Q)^{-1}$, where $\mathcal{D}$ is the Kähler different of $R$ over $Q$, i.e. the ideal generated by all $r \times r$ minors of $J$. In particular, we have an exact sequence

$$0 \to K_R \to \mathcal{D}(R/Q)^{-1} \to T(I/I^2) \to 0.$$

**Proof.** By tensoring the middle row of (4) with $L$ we see that $D \otimes_R L \cong L$. By Cramer's rule $\ker(\alpha \otimes L) = L \cdot (\Delta_1 e_1 + \cdots + \Delta_{r+1} e_{r+1})$, where $\Delta_1, \ldots, \Delta_{r+1}$ are the $r \times r$ minors of $J$ (with suitable signs). $D$ can be identified with the set of all $\lambda \in L$ for which $\lambda \Delta_i \in R$ ($i = 1, \ldots, r + 1$), i.e. with $\mathcal{D}(R/Q)^{-1}$.

**3. Applications to differential forms.** Under the assumptions as in the beginning of §2 we consider in the $L$-vector space $\Lambda^d(L \otimes_R D_k(R))$ of "meromorphic $d$-forms" the $R$-submodule

$$\Omega^d_R := \mathcal{D}(R/Q)^{-1} dx_1 \wedge \cdots \wedge dx_d.$$
Lemma 2. $\Omega_R'$ does not depend on the choice of $Q \subset R$.

Let $Q' = k[y_1, \ldots, y_d]$ be another subalgebra of $R$ having analogous properties as $Q$. In $\Lambda^d(L \otimes_R D_k(R))$ we have an equation

$$dx_1 \wedge \cdots \wedge dx_d = \delta dy_1 \wedge \cdots \wedge dy_d \quad (\delta \in L \setminus \{0\})$$

and in $\Lambda^n(L \otimes_S D_k(S))$

$$dx_1 \wedge \cdots \wedge ds_i \wedge dF_i \wedge \cdots \wedge dF_i$$

if $F_i, \ldots, F_i$ are taken from a set of generators $\{F_1, \ldots, F_m\}$ of $\mathcal{I}$. From this we can conclude that $\mathcal{O}(R/Q) = \delta \mathcal{O}(R/Q')$ and $\mathcal{O}(R/Q)^{-1}dx_1 \wedge \cdots \wedge dx_d = \mathcal{O}(R/Q')^{-1}dy_1 \wedge \cdots \wedge dy_d$.

Let $\mathfrak{C}(R/Q)$ be the Dedekind complementary module of $\mathfrak{A}$ over $Q$, i.e. the set of all $\mathfrak{A} \in L$ such that $\mathfrak{C}(R/Q) \mathfrak{A} \subset Q$ for all $r \in R$, where $\mathfrak{C}(R/Q)$ is the canonical trace. It is known that $\mathfrak{C}(R/Q) \subset \mathfrak{C}(R/Q)^{-1}$ and that $\mathfrak{C}(R/Q)$ is a canonical module of $R$. Moreover $\Omega_R := \mathfrak{C}(R/Q)^{-1}dx_1 \wedge \cdots \wedge dx_d$ does not depend on the choice of $Q$ (see [6]).

The $R$-modules $\Omega_R$ and $\Omega_R'$ represent two possibilities to define “regular $d$-forms for $R$”. A third one is given by taking the image $\Omega_R'$ of $\Lambda^dD_k(R)$ in $\Lambda^d(L \otimes_R D_k(R))$.

If $R$ is a regular local ring, then $\Omega_R = \Omega_R' = \Omega_R''$. For a complete intersection $R$ still $\Omega_R = \Omega_R''$. The situation for almost complete intersections describes

Theorem 2. Let $R$ be an almost complete intersection. Then

$$\Omega_R/\Omega_R \cong T(I/I^2)$$

Hence the following conditions are equivalent:

(a) $\Omega_R = \Omega_R'$.

(b) $R_P$ is a complete intersection for all $P \in \text{Spec}(R)$, $ht(P) = 1$.

(c) $\Omega_R$ is reflexive.

Proof. We shall use the construction of the exact sequence (1) given by Matsuoka [7]. By the “Primbasissatz” there is a minimal system of generators $\{F_1, \ldots, F_{r+1}\}$ of $I$ such that $\{F_1, \ldots, F_r\}$ is $S$-regular sequence and $I_S = \langle F_1, \ldots, F_r \rangle \cdot S$.

Let $J := \langle F_1, \ldots, F_r \rangle \cdot S$. Then $K_R \cong J: I/J$ as $R$-module. There is a well-defined map $\gamma: J: I/J \rightarrow R^{r+1}$ given as follows: For $G \in J: I$ let $-GF_{r+1} = G_1F_1 + \cdots + G_rF_r$ ($G_i \in S$). Then $\gamma$ maps $G$ onto $\Sigma_{i=1}^{r+1} G_ie_i + \Sigma_{i=1}^{r+1} G_ie_{i+1}$, where $G_i, G_i$ are the images of $G, G_i$ in $R$. $\gamma$ induces an injection $J: I/J \rightarrow R^{r+1}$ whose image is the kernel of $R^{r+1} \rightarrow I/I^2$.

Let $\Sigma = S/J$. We can choose $Q = k[x_1, \ldots, x_d]$ such that $\Sigma$ is a $Q$-module of finite type (and $L$ separable algebraic over $K$, as before). Since
$L$ is the residue field of $S_f$, we can conclude that
$$\Delta_{r+1} = \frac{\partial (F_1, \ldots, F_r)}{\partial (x_{d+1}, \ldots, x_n)} \neq 0.$$

We use diagram (4) with the sequence $0 \to K_r \to R^{r+1} \to I/I^2 \to 0$ as described above. With the notations as in the proof of Lemma 1 the mapping $D \to R^{r+1}$ identifies each $\lambda \in \mathfrak{D}(R/Q)^{-1}$ with $\lambda(\Delta_1 e_1 + \cdots + \Delta_r e_r) \in R^{r+1}$. This element is in $\ker(\beta)$ iff $\lambda \Delta_{r+1}$ is in the image of $J : I$ in $R$. In order to prove Theorem 2 it is therefore sufficient to show

**Lemma 3.** If $\Delta$ is the image of $J : I$ in $R$, then
$$\Delta = \frac{\partial (F_1, \ldots, F_r)}{\partial (x_{d+1}, \ldots, x_n)} \cdot \mathfrak{c}(R/Q).$$

**Proof.** Let $I' = J : I$. We have $J = I \cap I'$ and $I'$ has only associated primes $P_1, \ldots, P_s$ of height $r$ and different from $I$. If $\bar{I}, \bar{I'}$ and $\bar{P}_i$ $(i = 1, \ldots, s)$ denote the images in $\Sigma$, then $\bar{I} \cap \bar{I'} = (0)$, $\bar{I'}$ = Ann$_k(\bar{I})$ and $\Sigma_{\bar{I}} = L$. For the full ring of quotients of $\Sigma$ we have
$$Q(\Sigma) = K \otimes Q \cdot \Sigma = L \times \Sigma_{\bar{P}_1} \times \cdots \times \Sigma_{\bar{P}_s}. \quad (5)$$

The image of $\bar{I'}$ in $Q(\Sigma)$ is $\Delta \times (0) \times \cdots \times (0)$.

In the commutative diagram of canonical homomorphisms
$$\xymatrix{ \text{Hom}_Q(R, Q) \ar[r]^\alpha \ar[d]_{\beta} & \text{Hom}_K(L, K) \ar[d] \cr \text{Hom}_Q(\Sigma, Q) \ar[r] & \text{Hom}_K(K \otimes Q, \Sigma, K) \rlap{ all mappings are injective.} }$$

Let $\sigma_{L/K} : L \to K$, $\sigma_{\Sigma/Q} : \Sigma \to Q$ and $\sigma : K \otimes Q \Sigma \to K$ be the canonical traces. We have $\text{Hom}_K(L, K) = L \sigma_{L/K}$ and $\text{im}(\alpha) = \mathfrak{c}(R/Q) \sigma_{L/K}$. Moreover, since $\Sigma/Q$ is a complete intersection, $\text{Hom}_Q(\Sigma, Q) = \Sigma \cdot \eta$ with a trace map $\eta : \Sigma \to Q$, which by Scheja-Storch [8, 4.2], can be chosen in such a way that
$$\sigma_{\Sigma/Q} = \frac{\partial (F_1, \ldots, F_r)}{\partial (x_{d+1}, \ldots, x_n)} \cdot \eta,$$

where $\partial (F_1, \ldots, F_r)/\partial (x_{d+1}, \ldots, x_n)$ denotes the image of the Jacobian determinant in $\Sigma$.

We have $\text{im}(\beta) = \bar{I'} \eta$, since for $s \in \Sigma$ the map $s \eta$ factors through $R$ iff $s \in \bar{I'}$. From (5) we get a decomposition
$$\text{Hom}_K(K \otimes Q, \Sigma, K) = (L \times \Sigma_{\bar{P}_1} \times \cdots \times \Sigma_{\bar{P}_s}) \cdot \eta$$
$$= \text{Hom}_K(L, K) \times \text{Hom}_K(\Sigma_{\bar{P}_1}, K) \times \cdots \times \text{Hom}_K(\Sigma_{\bar{P}_s}, K),$$
and Hom\(_K(\Sigma, K) \to Hom\_K(K \otimes Q \Sigma, K)\) is the canonical injection onto the first factor.

The image of \(\overline{T} \eta\) in Hom\(_K(K \otimes Q \Sigma, K)\) is

\[
(\Delta \times (0) \times \cdots \times (0)) \cdot \eta = \left[\Delta_{r+1}^{-1} \cdot (\Delta \times (0) \times \cdots \times (0))\right] \cdot \sigma
= (\Delta_{r+1}^{-1} \cdot \Delta \cdot \sigma_{L/K}) \times (0) \times \cdots \times (0).
\]

We obtain \(\mathfrak{G}(R/Q) \cdot \sigma_{L/K} = \Delta_{r+1}^{-1} \cdot \Delta \cdot \sigma_{L/K},\) which proves the claim.

If \(R\) is an almost complete intersection of dimension 1 the length of \(T(I/I^2)\) is related to the length of the torsion \(T(D_k(R))\) of the differential module.

Consider the exact sequence

\[
0 \to T(D_k(R)) \to D_k(R) \to \Omega_R \to C \to 0,
\]

where \(C\) is the cokernel of the canonical map \(D_k(R) \to \Omega_R\).

If \(Q \subset R, Q = k[x]\) is chosen as above, then \(Rdx\) is a free submodule of \(D_k(R)\) and hence we have also exact sequences

\[
0 \to T(D_k(R)) \to D_k(R)/Rdx \to \Omega_R/Rdx \to C \to 0
\]

and

\[
0 \to \Omega_R/Rdx \to \Omega_R/Rdx \to T(I/I^2) \to 0.
\]

This gives us the length-formula

\[
l(C) = l(T(D_k(R))) - l(T(I/I^2)) + l(\Omega_R/Rdx) - l(D_k(R)/Rdx).
\]

By Berger [2, Satz 2], \(l(D_k(R)/Rdx) = l(\mathfrak{G}(R/Q)^{-1}/R) = l(\Omega_R/Rdx),\) therefore

\[
l(C) = l(T(D_k(R))) - l(T(I/I^2)).
\]

Since \(\dim R = 1\) we have \(T(I/I^2) \neq 0\), hence

\[
0 < l(T(I/I^2)) < l(T(D_k(R)))
\]

and

\[
l(C) < l(T(D_k(R))).
\]

References

ALMOST COMPLETE INTERSECTION


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