EXISTENCE OF FIXED POINTS OF NONEXPANSIVE MAPPINGS IN CERTAIN BANACH LATTICES

PAOLO M. SOARDI

Abstract. A fixed point theorem for nonexpansive mappings in dual Banach spaces is proved. Applications in certain Banach lattices are given.

1. Suppose $K$ is a subset of a Banach space $X$ and $T: K \to K$ is a nonexpansive mapping, i.e. $\|T(x) - T(y)\| \leq \|x - y\|$, $x, y \in K$. A well-known theorem due to Kirk [1] states that, if $K$ is convex weakly compact (weak* compact when $X$ is a dual space) and has normal structure, then $T$ has a fixed point in $K$. In particular, if $X = L^p$ ($1 < p < \infty$) Kirk's theorem applies to every bounded closed convex set $K$, while an analogous theorem was proved by Karlovitz [3, Corollary] in $l^1$. No result seems to be known in $L^\infty$.

In this paper we study the existence of fixed points of nonexpansive mappings in certain (complex) AM-spaces. First we prove a general fixed point theorem for nonexpansive mappings in dual spaces and then we draw some consequences for nonexpansive mappings acting in (complex) AM-spaces which are dual to (complex) AL-spaces. These results imply, for instance, that every nonexpansive operator mapping into itself a closed ball $B \subseteq L^\infty$ has a fixed point in $B$, and every nonexpansive $T: L^\infty \to L^\infty$, which leaves invariant a weak* compact subset, has a fixed point (in $L^\infty$).

2. A real Banach lattice $X$ is called an AM-space (abstract-m-space) if $\|x \vee y\| = \|x\| \vee \|y\|$, for every $x, y \in X$ such that $x, y \geq 0$. Here and in the sequel $\vee$ and $\wedge$ denote the least upper bound and the greatest lower bound respectively. $X$ is said to be order complete if each set $A \subseteq X$ with an upper bound has a least upper bound. A complex AM-space is defined as the complexification of an AM-space.

Suppose $X$ is an order complete AM-space with unit (i.e. an element $e$ such that the unit ball at zero is the order interval $[-e, e]$); then $X$ is isometrically lattice isomorphic to the space $C_e(S)$ of all continuous real-valued functions defined on a compact Stonian space $S$.

For these and other facts about Banach lattices we refer to Schaefer’s book [4].
The following notation will be used throughout the paper: for a Banach space $X$, $B(x, r)$ denotes the closed ball centered at $x \in X$ of radius $r$; if $M \subseteq X$ is a nonvoid bounded subset, diam $M$ denotes the diameter of $M$ and $\text{co} M$ the closed convex hull of $M$.

**Lemma.** Suppose $X$ is the complexification of an order complete AM-space with unit. For every nonvoid bounded closed set $M \subseteq X$ there exists a point $z_M \in X$ with the following properties:

(a) $M \subseteq B(z_M, 2^{-1/2} \text{ diam } M)$.

(b) If, for some $y \in X$, $M \subseteq B(y, 2^{-1/2} \text{ diam } M)$, then $\|z_M - y\| < 2^{-1/2} \text{ diam } M$.

A similar statement holds if $X$ is an order complete AM-space with unit, with $2^{-1/2}$ replaced by $1/2$.

**Proof.** We shall prove the lemma in the complex case, the real one being simpler. Let $C(S)$ denote the space of continuous complex-valued functions which represents $X$. Set:

$$a_1 = \sqrt{\int_M \Re f}, \quad a_2 = \sqrt{\int_M \Re f}, \quad a_3 = \sqrt{\int_M \Im f}, \quad a_4 = \sqrt{\int_M \Im f}.$$

By our assumption on $X$ the $a_j$'s are continuous real-valued functions belonging to $C_R(S)$. Let $u = (a_1 + a_2)/2$, $v = (a_3 + a_4)/2$. We shall prove that $z_M = u + iv$ has the desired properties. Indeed, if $r = \text{diam } M$, it is easy to see that $\|a_1 - a_2\| < r$ and $\|a_3 - a_4\| < r$ so that, for every $s \in S$ and $f \in M$, $|\Re f(s) - u(s)| < r/2$, $|\Im f(s) - v(s)| < r/2$; hence (a) holds.

Suppose now that $M \subseteq B(y, 2^{-1/2} \text{ diam } M)$ for some $y \in X$. Let $\epsilon > 0$ be arbitrarily small. For every $s \in S$ we can find a neighborhood of $s$, $V(s)$, such that $|a_j(t_1) - a_j(t_2)| < \epsilon$ ($j = 1, 2, 3, 4$) and $|y(t_1) - y(t_2)| < \epsilon$ whenever $t_1, t_2 \in V(s)$.

Since $S$ is extremally disconnected, we have:

$$a_1(s) = \inf_{U(s)} \sup_{t \in U(s)} \sup_{f \in M} \Re f(t)$$

where $U(s)$ runs through a neighborhood base of $s$ (see [4, p. 107]). Hence there is a point $s_1 \in V(s)$ and a function $f_{1,s} \in M$ such that: $|\Re f_{1,s}(s_1) - a_1(s)| < \epsilon$. Therefore

$$|a_1(s) + i \Re f_{1,s}(s_1) - y(s)| < 2\epsilon + 2^{-1/2}\epsilon.$$

Analogously, we can find points $s_j \in V(s)$ and functions $f_{j,s} \in M$ such that $|g_j(s) - y(s)| < 2\epsilon + 2^{-1/2}\epsilon$, where

$$g_j(s) = a_j(s) + i \Re f_{j,s}(s_j) \quad \text{if } j = 1, 2,$$

$$g_j(s) = \Re f_{j,s}(s_j) + i a_j(s) \quad \text{if } j = 3, 4.$$

Since the oscillation of the $a_j$'s on $V(s)$ is less than $\epsilon$, an elementary geometric argument shows that there is a number in the convex hull of the $g_j(s)$'s whose distance from $z_M(s)$ is less than $\epsilon$. Therefore $|z_M(s) - y(s)| < 3\epsilon + 2^{-1/2}\epsilon$ for all $s \in S$, whence (b).
**Remark.** It follows from the above construction that if $B$ is a closed ball containing $M$, then $z_M \in B$. Analogously, if $M$ is contained in some order interval $I = \{x \in X: a < x < b\}$, then $z_M \in I$ (in the case $X$ is an AM-space). Therefore we are led to the following definition.

**Definition.** A closed subset $K$ of a Banach space $X$ has uniform relative normal structure if there exists $c < 1$ such that, for every nonvoid bounded closed subset $M \subseteq K$, there exists $z_M \in K$ with the following properties:

(a') $M \subseteq B(z_M, c \text{ diam } M)$.
(b') If, for some $y \in K$, $M \subseteq B(y, c \text{ diam } M)$, then $\|z_M - y\| < c \text{ diam } M$.

This definition should be compared with the analogous definition in [2].

**Theorem.** Suppose $X$ is a dual Banach space and $K \subseteq X$ a weak* closed set with uniform relative normal structure. Let $T: K \to K$ be a nonexpansive mapping which leaves invariant a weak* compact subset $M \subseteq K$ (i.e. $T(M) \subseteq M$). Then there exists $u \in K$ such that $u = T(u)$.

**Proof.** Let $A_0 \subseteq M$ be minimal among weak* compact invariant subsets of $M$. Then, if $\text{cl}^*$ denotes the weak* closure,

$$T(\text{cl}^* T(A_0)) \subseteq T(A_0) \subseteq \text{cl}^* T(A_0)$$

so that $A_0 = \text{cl}^* T(A_0)$. Suppose $r$ is the diameter of $A_0$. Then the set $A = \{z \in K: A_0 \subseteq B(z, cr)\}$ is nonvoid, since $z_{A_0} \in A$. Moreover $A$ is weak* compact, as an intersection of closed balls with $K$. Fix $\epsilon > 0$ arbitrarily. For every $z \in A$ and $x \in A_0$ there exists $y \in A_0$ such that:

$$\|T(z) - x\| - \epsilon < \|T(z) - T(y)\| < \|z - y\| < cr.$$

Since $\epsilon$ is arbitrary, $\|T(z) - x\| < cr$ and $T(A) \subseteq A$.

Define a set $H$ by $H = \{z \in A: A \subseteq B(z, cr)\}$. $H$ is nonvoid since $z_{A_0} \in H$. Let $A_1$ denote the intersection of all $w^*$ compact invariant subsets of $A$ containing $H$. An argument due essentially to Kirk [1] shows that $\text{diam } A_1 < cr$. Namely, let $F$ denote the set $\{z \in A: A_1 \subseteq B(z, cr)\}$. $F$ contains $H$ and is weak* compact (as an intersection of closed balls with $A_1$). Assume, by way of contradiction, that $T(z) \notin F$ for some $z \in F$; then the set $G = B(T(z), cr) \cap A_1$ is weak* compact and contains $H$. Moreover, for every $x \in G$: $\|T(z) - T(x)\| < \|z - x\| < cr$, since $z \in F$. Hence $T(G) \subseteq G$ and, by the definition of $A_1$, $A_1 = G$. But $T(z) \notin F$, so that $\|T(z) - x\| > cr$ for some $x \in A_1$, a contradiction. Consequently $T(F) \subseteq F$, and by the definition of $A_1$ again, $F = A_1$. Hence $\text{diam } A_1 = \text{diam } F < cr$. Moreover $A_0 \subseteq B(x, cr)$ for every $x \in A_1$. Repeating this construction, we define inductively a sequence of weak* compact subsets $A_n \subseteq K$ with the properties:

(i) $\text{diam } A_n < rc^n$,
(ii) $T(A_n) \subseteq A_n$,
(iii) $\|x - y\| < rc^n$, whenever $x \in A_n, y \in A_{n-1}$. 

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If we pick a sequence of points \( u_n \in A_n \) we have:
\[
\|u_n - u_m\| < r \sum_{n=0}^{m} c^n \quad (n < m) \quad \text{and} \quad \|u_n - T(u_n)\| < rc^n.
\]
Therefore \( u_n \) converges in the norm topology to a point \( u \in K \) such that \( u = T(u) \).

We recall that an AL-space is a real Banach lattice such that \( \|x + y\| = \|x\| + \|y\| \) whenever \( x, y \geq 0 \). A complex AL-space is defined to be the complexification of a real AL-space. It is known [4] that the dual of a (complex) AL-space is a (complex) AM-space which satisfies the assumptions of the above lemma. Henceforth we have the following consequences.

**Corollary 1.** Suppose \( X \) is the dual of a (complex) AL-space. Then, if \( B \subseteq X \) is a closed ball and \( T: B \to B \) is a nonexpansive mapping, \( T \) has a fixed point in \( B \).

**Corollary 2.** Suppose \( X \) is the dual of an AL-space. If \( I \subseteq X \) is a closed order interval and \( T: I \to I \) is a nonexpansive mapping, \( T \) has a fixed point in \( I \).

**Corollary 3.** Suppose \( X \) is the dual of a (complex) AL-space. If \( T: X \to X \) is a nonexpansive mapping which leaves invariant a weak* compact subset of \( X \), \( T \) has a fixed point (in \( X \)).

**Remark.** Suppose \( (Y, \Sigma, \mu) \) is a \( \sigma \)-finite measure space; the dual of the (complex) AL-space \( L^1(Y, \Sigma, \mu) \) is identified with \( L^\infty(Y, \Sigma, \mu) \), so that the above corollaries hold with \( L^\infty(Y, \Sigma, \mu) \) in place of \( X \).

3. In this section we give an example of uniform relative normal structure in spaces which are not AM-spaces.

Suppose \( X \) is a uniformly convex Banach space; denote by \( X^* \) its dual space and by \( \|\cdot\| \) and \( \|\cdot\|^* \) the norms in \( X \) and \( X^* \) respectively. Let \( Z \) denote the space of all sequences \( z = (z_1, z_2, \ldots, z_n, \ldots) \), \( z_n \in X \), such that \( \sup_n \|z_n\| = \|z\|_\infty < \infty \). \( Z \) is not an AM-space unless \( X \) itself is an AM-space. Moreover \( Z \) is the dual of the space of all sequences \( t = (t_1, t_2, \ldots, t_n, \ldots) \), \( t_n \in X^* \), such that \( \Sigma_n \|t_n\|^* < \infty \).

**Proposition.** \( Z \) and every closed ball \( B \subseteq Z \) have uniform relative normal structure.

**Proof.** The proof is achieved by generalizing the argument used in the lemma of §2. Suppose \( C \subseteq Z \) is a closed nonvoid bounded set. Let \( z = (z_1, z_2, \ldots, z_n, \ldots) \) belong to \( C \) and denote by \( C_n \) the subset of \( X \) described by \( z_n \) as \( z \) describes \( C \). Since \( X \) is uniformly convex, there exists \( c < 1 \), independent from \( C \) and \( n \), such that there exist points \( z_{C,n} \in \overline{C_n} \) with the property \( \|z_{C,n} - z_n\| < cr \) for every \( z_n \in C_n \), \( n = 0, 1, 2, \ldots \), (here we made \( r = \text{diam } C \)). Thus the point \( z_C = (z_{C,1}, \ldots, z_{C,n}, \ldots) \) has the property (a').

On the other hand suppose that the point \( y = (y_1, y_2, \ldots, y_n, \ldots) \) is such that \( C \subseteq B(y, cr) \). It follows that \( C_n \subseteq B(y_n, cr) \) for every \( n \). By the
definition of $z_{C,n}$, we have $z_{C,n} \in B(y_n, cr)$ too, so that $\|z_C - y\|_\infty \leq cr$. It is also clear that $z_C \in B$ if $C$ is contained in the closed ball $B$.

From this proposition it is possible to deduce the analogues of Corollaries 1 and 3.

REFERENCES


ISTITUTO MATEMATICO DELL'UNIVERSITÀ, VIA SALDINI 50, 20133 MILANO, ITALY