ON A CONJECTURE OF HUNT AND MURRAY CONCERNING 
q-PLURISUBHARMONIC FUNCTIONS

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ABSTRACT. We discuss the conjecture of Hunt and Murray on uniqueness of 
the Dirichlet problem for the generalized complex Monge-Ampère equation. 
We define a class of q-plurisubharmonic functions, prove uniqueness in this 
class, and show that in some cases the solution found by Hunt and Murray 
is in our class.

method [3] of solving the Dirichlet problem for generalized complex Monge-
Ampère equation to a class of q-plurisubharmonic functions, Pq. They 
conjecture that their extremal function, \( \tilde{u}(z) = \sup \{ v(z) : v \in P_q, v < b \text{ on } \partial \Omega \} \), is the unique solution of the problem, \( u \in P_q(\Omega) \cap (-P_{n-q-1}(\Omega)), u \in C(\bar{\Omega}), u | \partial \Omega = b(z), \text{ in the class } P_q \). What we do here is define a 
somewhat smaller class \( \tilde{P}_q \). We show that for this class the Dirichlet problem 
above indeed has a unique solution. We show further that in certain special 
cases \( \tilde{u}(z) \) actually solves the Dirichlet problem for \( \tilde{P}_q \) and hence is the 
unique solution, in \( \tilde{P}_q \). In our discussion we need to assume that \( 2q < n \) and 
\( \partial \Omega \) is strictly q-pseudoconvex-exactly as in [4].

1. \( \tilde{P}_q(\Omega) \). Let \( \Omega \subseteq C^n \) be a domain. We begin by recalling the definition of 
q-plurisubharmonic function as found in [4].

(1.1) DEFINITION. (A) \( u : \Omega \to \mathbb{R} \cup \{-\infty\} \) is said to be \((n-1)\) plurisub-
harmonic if

(1) \( u \) is upper semicontinuous on \( \Omega \).

(2) If \( B \subset \Omega \) is a ball and \( g \) is a lower semicontinuous plurisuperharmonic 
function on \( B \), then \( g \geq u \) on \( \partial B \Rightarrow g \geq u \) on \( B \).

(B) \( u \) is said to be \( q \)-plurisubharmonic if \( u \) is \( q \)-plurisubharmonic on 
\( \Omega \cap \Pi_{q+1} \), where \( \Pi_{q+1} \) is a complex linear \((q+1)\) dimensional subspace of 
\( C^n \), with \( \Pi_{q+1} \cap \Omega \neq \emptyset \).

We will use the notation \( P_q(\Omega) \) for the class of all \( q \)-plurisubharmonic 
functions on \( \Omega \), and \( PS_q(\Omega) = -P_q(\Omega) \) for the class of \( q \)-plurisuperharmonic 
functions. We also note that since plurisuperharmonic functions are the 
increasing limit of smooth plurisuperharmonic functions, it is equivalent to 
define \( P_q(\Omega) \) as functions (upper semicontinuous) which satisfy a maximum

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principle with respect to smooth plurisuperharmonic functions.

We now define another class of $q$-plurisubharmonic functions, contained in $P_q(\Omega)$. It is for this class of functions that we are able to prove uniqueness in the Dirichlet problem.

(1.2) Definition. (A) Let $\tilde{P}_q(\Omega)$ denote the class of functions which are upper semicontinuous on $\Omega$ and which satisfy the following condition: if $B \subset \Omega$ is a ball and $g \in PS_{n-q-1}(\Omega)$ satisfies $g \geq u$ on $\partial B$, then $g \geq u$ on $B$.

(B) We define $\tilde{PS}_q(\Omega) = -\tilde{P}_q(\Omega)$ as functions satisfying a minimum principle with respect to $P_{n-q-1}(\Omega)$.

(1.3) Proposition. $\tilde{P}_q(\Omega) \subset P_q(\Omega)$.

Proof. The idea here is the same as the first part of the proof of Theorem 3.3 of [4]. Let $u \in \tilde{P}_q(\Omega)$ and let $\Pi_{q+1}$ be a $(q + 1)$ dimensional complex plane intersecting $\Omega$. After a linear change of coordinates we can assume that $\Pi_{q+1} = \{z^1 = \ldots = z^{n-q-1} = 0\}$. Let $g \in PS_0(\Pi_{q+1})$ be such that $g \geq u$ on $\partial B \cap \Pi_{q+1}$, where $B \subset \Omega$ is a ball. We extend $g$ to $B$ by $\tilde{g} = f(\sum_{j=1}^{q-1}|z_j|^2) + g(z^{n-q}, \ldots, z^n)$. It is easy to see that $\tilde{g}$ is in $PS_{n-q-1}(\Omega)$. We choose $f$ so that $f$ is lower semicontinuous, $f(0) = 0$ and $\tilde{g} \geq u$ on $\partial B$. Then $\tilde{g} \geq u$ on $B$.

(1.4) Remarks. (1) It is easy to see that $\tilde{P}_q(\Omega) = P_q(\Omega)$ is just the class of plurisubharmonic functions, and that $\tilde{P}_{n-1}(\Omega) = P_{n-1}(\Omega)$.

(2) It seems unlikely to us that $\tilde{P}_q(\Omega) = P_q(\Omega)$ for $0 < q < n - 1$.

Even if we define $\tilde{P}_q(\Omega)$ as those functions satisfying a maximum principle with respect to smooth $(n - q - 1)$ plurisuperharmonic functions, it seems that one would have to know that such functions are plurisuperharmonic functions on some $(q + 1)$ dimensional local complex analytic submanifold through every point. This is not true, for example if the complex Hessian does not have constant rank and has at least $(q + 1)$ eigenvalues equal to zero at every point (for a discussion of this see [2]). Even if the complex Hessian does have constant rank but there are some positive eigenvalues Eric Bedford has communicated a counterexample to us, based on the function given on the bottom half of p. 4.11 of [1].

2. The generalized Dirichlet problem.

(2.1) Definition. We say that $u \in \tilde{P}_q(\Omega)$ satisfies the generalized complex Monge-Ampère equation in $\tilde{P}_q$ if $u \in \tilde{P}_q(\Omega) \cap \tilde{PS}_{n-q-1}(\Omega)$.

We note that if $u$ satisfies the generalized complex Monge-Ampère equation then $u$, being both upper and lower semicontinuous, is continuous. Now if $u \in C^2(\Omega) \cap \tilde{P}_q(\Omega) \cap \tilde{PS}_{n-q-1}(\Omega)$ then in particular $u \in C^2(\Omega) \cap P_q(\Omega) \cap PS_{n-q-1}(\Omega)$. In [4] it is shown that $C^2(\Omega) \cap P_q(\Omega) = \{u \in C^2(\Omega): dd^cu(p) \text{ has at least } (n-q) \text{ nonnegative eigenvalues for all } p\}$, and $C^2(\Omega) \cap PS_{n-q-1}(\Omega) = \{u \in C^2(\Omega): dd^cu(p) \text{ has at least } (q+1) \text{ nonpositive eigenvalues for all } p\}$. So if $u$ satisfies the generalized Monge-Ampère equation in
\( P_q \) and is in \( C^2(\Omega) \), then \( dd^cu(p) \) must have at least one zero eigenvalue, so
\[
\underbrace{dd^cu \wedge \cdots \wedge dd^cu}_{n\text{-times}} = 0.
\]

We cannot characterize \( C^2(\Omega) \cap \tilde{P}_q \) quite so simply as was done in [4]. However we can prove the following result.

(2.2) **Proposition.** Let \( \mathcal{H}^+_n = \{ u \in C^2(\Omega) \colon dd^cu(p) \text{ has at least } (n - q) \text{ positive eigenvalues at each point } p \in \Omega \} \). Then we have \( \mathcal{H}^+_n \subset \tilde{P}_q(\Omega) \).

**Proof.** To prove this result all we need note is that we have only to prove the result locally. Now, working at a fixed point \( 0 \in \Omega \), we can assume that \( dd^cu(0) \) is positive definite on the plane \( \Pi = \{ z^{n-q-1} = \cdots = z^n = 0 \} \). Then in a neighborhood of \( 0 \), \( u \) is plurisubharmonic on \( \Pi \). Then if \( B \subset \Pi \cap \Omega \) is a ball and \( g \in PS_{n-q-1}(\Pi) \) is such that \( g > u \) on \( \partial B \) we have that \( g > u \) on \( B \).

In order to prove uniqueness of the Dirichlet problem we will need the following result:

(2.3) **Proposition.** \( \tilde{P}_q(\Omega) + \tilde{P}_q(\Omega) = \tilde{P}_{n-1}(\Omega) = P_{n-1}(\Omega) \) if \( q_1 + q_2 = n - 1 \).

Before we prove this we give the following lemma needed in the proof.

(2.4) **Lemma.** \( \tilde{P}_0(\Omega) + \tilde{P}_q(\Omega) \subset \tilde{P}_q(\Omega) \).

**Proof.** Suppose \( g \in \tilde{P}_0, u \in \tilde{P}_q \) and \( h \in PS_{n-q-1} \) are such that \( h > g + u \) on \( \partial B \) where \( B \) is a ball in \( \Omega \). Then it suffices to show that \( h - g \in PS_{n-q-1} \). Let \( \Pi_{n-q} \) be an \((n - q)\) complex dimensional plane such that \( \Pi_{n-q} \cap \Omega \neq \emptyset \). Then let \( k \in P_0(\Pi_{n-q}) \), \( k < h - g \) on \( \partial B' \) where \( B' \) is a ball in \( \Pi_{n-q} \cap \Omega \). Then \( k + g \in P_0(\Pi_{n-q}) \) so \( k + g < h \) in \( B' \).

Now we are able to give the proof of Proposition 2.3.

**Proof.** Let \( u_1 \in P_q(\Omega), i = 1, 2 \). Let \( g \in PS_0 \) be such that \( g > u_1 + u_2 \) on \( \partial B \); where, again, \( B \subset \Omega \) is a ball. Then \( g - u_2 \in PS_{n-q-1} \) by Lemma 2.4 and \( g - u_2 > u_1 \) on \( \partial B \). So the definition of \( \tilde{P}_q \) implies that \( u_1 < g - u_2 \) on \( B \).

(2.5) **Theorem.** Suppose \( u_1, u_2 \in C(\Omega) \cap \tilde{P}_q(\Omega) \) are both solutions of the generalized complex Monge-Ampère equation in \( P_q(\Omega) \), and further suppose \( u_1 \mid \partial \Omega = u_2 \mid \partial \Omega \). Then \( u_1 = u_2 \) in \( \Omega \).

**Proof.** \( u_1 \in \tilde{P}_q(\Omega) \cap \tilde{PS}_{n-q-1}(\Omega) \). We will use Proposition 2.3. \( u_1 - u_2 \in \tilde{P}_q(\Omega) + \tilde{PS}_{n-q-1} \subset \tilde{P}_{n-1}(\Omega) \) since \( u_2 \in \tilde{PS}_{n-q-1}(\Omega) \). This gives that \( u_1 - u_2 < 0 \) in \( \Omega \). But on the other hand \( u_2 - u_1 \in \tilde{P}_q(\Omega) - \tilde{P}_{n-1}(\Omega) \) since \( u_1 \in PS_{n-q-1}(\Omega) \). So we have \( u_2 - u_1 < 0 \) on \( \Omega \).

Let \( \Omega \subset C^n \) be a strictly \( q \)-pseudoconvex domain with smooth boundary. We consider the following generalized Dirichlet problem for the complex
Monge-Ampère equation. For $b(z) \in C(\partial \Omega)$, let
\[ \mathcal{B}_q(b, \Omega) = \{ v \in P_q(\Omega): u(z) < b(z) \text{ on } \partial \Omega \}. \]

Let $u(z) = \sup_{v \in \mathcal{B}_q(b, \Omega)} v(z)$. Then if $2q < n$ it is shown in [4] that
(1) $u(z)|\partial \Omega = b(z)$.
(2) $u(z) \in C(\Omega)$.
(3) $u(z) \in P_q(\Omega) \cap PS_n^{m-q-1}(\Omega)$.

Now our uniqueness Theorem 2.5 holds only if $u \in \check{P}_q(\Omega) \cap \check{PS}_n^{m-q-1}(\Omega)$.

We show now that in two (very special) cases, this is indeed the case.

(2.6) Theorem. Let $\beta \subset C^{2n+1}$ be a strictly $n$-pseudoconvex with smooth boundary. If $u(z) = \sup_{v \in \mathcal{B}_q(b, \Omega)} v(z)$, we have that $u(z)$ satisfies the generalized complex Monge-Ampère equation in $\check{P}_q(\Omega)$, i.e. $u \in \check{P}_q(\Omega) \cap \check{PS}_n(\Omega)$.

Proof. If we examine the proof of Theorem 3.3 of [4] we note that what is actually being proved there is that $u(z) \in \check{PS}_m^{n-q-1}(\Omega)$ for $\Omega \subset C^n$, $\Omega$ strictly $q$-pseudoconvex in $C^n$ with smooth boundary. Now we consider the lower envelope problem dual to the one defining $u(z)$. Let $\mathcal{B}_m^{n-q-1}(b, \Omega)$ denote the class of functions in $PS_m^{n-q-1}$ which are greater than or equal to $b(z)$ on $\partial \Omega$.

Then if $\Omega$ is strictly $(m-q-1)$ pseudoconvex and $2(m-q-1) < m$ we get a function $\check{u}^\prime$ defined by
\[ \check{u}^\prime(z) = \inf_{v \in \mathcal{B}_m^{n-q-1}(b, \Omega)} v(z). \]

Using the arguments of [3], [4], it is easy to see that $\check{u}^\prime(z)$ satisfies
(1) $\check{u}^\prime(\Omega) \subset PS_m^{n-q-1}(\Omega) \cap \check{P}_q$.
(2) $\check{u}^\prime \in C(\Omega)$.
(3) $\check{u}^\prime|\partial \Omega = b$.

Now in the case where $m = 2n + 1$, $q = n$, and $\Omega$ is strictly $n$-pseudoconvex with smooth boundary both hypotheses (for $\mathcal{B}_q$ and for $\mathcal{B}_m^{n-q-1}$) are satisfied and we get two functions $\check{u}(z)$ and $\check{u}^\prime(z)$. We claim that $\check{u}(z) = \check{u}^\prime(z)$.

To see this, firstly we note that $\check{u}^\prime > \check{u}$ on $\partial \Omega$. So since $\check{u}^\prime \in \check{PS}_n^{m-q-1}(\Omega)$ and $\check{u} \in P_q(\Omega)$ we get $\check{u}^\prime > \check{u}$ in $\Omega$. Similarly $\check{u}^\prime < \check{u}$ on $\partial \Omega$, and $\check{u} \in \check{P}_q$ and $\check{u} \in PS_n^{m-q-1}$ so $\check{u}^\prime < \check{u}$ in $\Omega$. So $\check{u} = \check{u}^\prime$ and $\check{u} \in \check{P}_q(\Omega) \cap \check{PS}_n(\Omega)$.

Now if $\Omega$ is strictly $q$-pseudoconvex with smooth boundary we can form the class
\[ \mathcal{B}_q(b, \Omega) = \{ u \in \check{P}_q(\Omega): u(z) < b(z) \text{ on } \partial \Omega \}. \]

Then as in Bremermann [3], we can form the upper envelope $\check{u}(z)$ defined by $\check{u}(z) = \sup_{v \in \mathcal{B}_q(b, \Omega)} v(z)$ and ask whether $\check{u}(z)$ satisfies the generalized complex Monge-Ampère equation for $\check{P}_q(\Omega)$. We do not know the answer to this in general but in case $\check{u}(z)$ satisfies the complex Monge-Ampère equation in $\check{P}_q(\Omega)$ we get the following result.

(2.7) Proposition. If $\check{u}$ satisfies the complex Monge-Ampère equation for $\check{P}_q(\Omega)$, then $\check{u}(z) = u(z)$.
Proof. Since $P_q(\Omega) \supset P_q(\Omega)$, it is clear that $\bar{u}(z) < \bar{u}(z)$. Now since $\bar{u} \in \bar{P}_q(\Omega)$ we get that $\bar{u} \in \bar{B}_q(b, \Omega)$, so $\bar{u}(z) = \sup_{v \in \bar{B}_q(b, \Omega)} v(z) \geq \bar{u}(z)$.

There is another instance when we can conclude that $\bar{u}$ satisfies that complex Monge-Ampère equation for $\bar{P}_q(\Omega)$. Suppose $\Omega$ is strictly pseudoconvex with smooth boundary. Then if we examine the proof of Theorem 3.2 of [4], we note that in this case it is not necessary to hypothesize that $2q < n$. Thus we get the following result.

(2.8) Proposition. If $\Omega$ is strongly pseudoconvex with smooth boundary then $\bar{u}$, the upper envelope of elements in $P_q(\Omega)$ which are $< b(z)$ on $\partial \Omega$, is in $\bar{P}_q(\Omega) \cap \bar{P}_{n-q-1}(\Omega)$, and is the unique solution of the generalized complex Monge-Ampère equation in $\bar{P}_q(\Omega)$, and is equal to $\bar{u}$, the upper envelope of elements of $\bar{P}_q(\Omega)$ which are $< b(z)$ on $\partial \Omega$.

Bibliography

1. E. Bedford and D. Burns, Holomorphic mappings of annuli in $\mathbb{C}^n$ and the associated extremal function, preprint.

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