THAT QUASINILPOTENT OPERATORS ARE NORM-LIMITS OF NILPOTENT OPERATORS REVISITED

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Abstract. A new short proof is given that every quasinilpotent operator on a separable, infinite dimensional, complex Hilbert space is a norm-limit of nilpotent operators.

Let $H$ be a separable, infinite dimensional, complex Hilbert space, and let $L(H)$ denote the algebra of all bounded linear operators on $H$.

In [2] it was shown that every quasinilpotent operator $T$ in $L(H)$ (i.e., every $T$ whose spectrum is the singleton $\{0\}$) is a norm-limit of nilpotent operators. The purpose of this note is to give a very short and easily remembered proof of this theorem. Our proof has two basic ingredients: the use of a model for quasinilpotent operators constructed in [4], and the use of a deep theorem of Voiculescu [7]. For completeness we review the relevant facts from [4] and [7] that we shall need.

Let $H_\infty = H \oplus H \oplus \ldots$ be the direct sum of $S_0$ copies of $H$ indexed by the positive integers, let $\kappa = (\kappa_n)_{n=1}^{\infty}$ be a monotone decreasing sequence of nonnegative numbers converging to zero, and let $U_\kappa$ denote the quasinilpotent backward weighted shift operator in $L(H_\infty)$ defined by the equation

$$U_\kappa(x_1, x_2, \ldots, x_n, \ldots) = (\kappa_1 x_2, \kappa_2 x_3, \ldots, \kappa_n x_{n+1}, \ldots).$$

Theorem A [4]. If $T$ is any quasinilpotent operator in $L(H)$, then there is a sequence $\kappa = \kappa(T) = (\kappa_n)_{n=1}^{\infty}$ with the above described properties and a subspace $M$ of $H_\infty$ that is invariant under the quasinilpotent operator $U_\kappa$ such that $T$ is similar to $U_\kappa|_M$ and such that the subspace $H_\infty \ominus M$ is infinite dimensional.

We shall need some lemmas, the first two of which are completely elementary.

Lemma 1. Let $T_1$ and $T_2$ belong to $L(H_1)$ and $L(H_2)$, respectively, and suppose that there exists a sequence $\{S_n\}_{n=1}^{\infty}$ of invertible operators $S_n: H_1 \to H_2$ such that $\|S_n^{-1} T_1 S_n - T_2\| \to 0$. If $T_1$ is a norm-limit of nilpotent operators, then so is $T_2$.

Lemma 2. The operator $U_\kappa$ in (1) is a norm-limit of nilpotent operators in $L(H_\infty)$ (since the sequence $\kappa$ converges to zero).

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The following lemma is only slightly less elementary and is a special case of [6, Proposition 6.18].

**Lemma 3.** Let $S = (S_{ij})_{i,j=1}^n$ be an $n \times n$ matrix with operator entries acting on the direct sum $\mathcal{H} \oplus \cdots \oplus \mathcal{H}$ of $n$ copies of $\mathcal{H}$ and having the property that $S_{ij} = 0$ if $i > j$. If each $S_{ii}, 1 \leq i \leq n,$ is a norm-limit of nilpotent operators, then so is $S$. On the other hand, if $S$ is a norm-limit of nilpotent operators, then so is $S_{11} \oplus \cdots \oplus S_{nn}$.

**Proof.** For $1 \leq i \leq n$, let $N_{ii}$ be a nilpotent operator such that $\|N_{ii} - S_{ii}\| < \varepsilon$. Then the matrix $\tilde{S}$ obtained by replacing each diagonal entry $S_{ii}$ by the corresponding operator $N_{ii}$ satisfies $\|	ilde{S} - S\| < \varepsilon$, and it is easy to see that $\tilde{S}$ is nilpotent. To go the other way, observe that by choosing $n$ sufficiently large, the matrix

$$\text{Diag}(1, n, \ldots, n^{n-1})(S_{ij})\text{Diag}(1, \frac{1}{n}, \ldots, \frac{1}{n^{n-1}})$$

can be made as close as desired to $S_{11} \oplus \cdots \oplus S_{nn}$, and use Lemma 1.

**Lemma 4.** If $T$ belongs to $\mathcal{L}(\mathcal{H})$ and is quasinilpotent, and Theorem A is used to write the space $\mathcal{H}_{\infty}$ as $\mathcal{H}_{\infty} = \mathcal{M} \oplus \mathcal{M}^+$ where $U_\mathcal{M} \mathcal{M} \subset \mathcal{M}$ and $T$ is similar to $T' = U_\mathcal{M}^T \mathcal{M}$, then in the corresponding matrix decomposition

$$U_\mathcal{M} = \begin{pmatrix} T' & A \\ 0 & S \end{pmatrix},$$

(2)

the operator $S$ is quasinilpotent and $T' \oplus S$ is a norm-limit of nilpotent operators.

**Proof.** That $S$ is quasinilpotent follows from the fact that $U_\mathcal{M}$ is quasinilpotent (Theorem A) and the form of the matrix (2). That $T' \oplus S$ is a norm-limit of nilpotent operators is an immediate consequence of Lemmas 2 and 3.

**Theorem B [1].** If $T$ is any quasinilpotent operator in $\mathcal{L}(\mathcal{H})$, then $T \oplus 0$ is a norm-limit of nilpotent operators in $\mathcal{L}(\mathcal{H} \oplus \mathcal{H})$.

**Proof.** According to Lemmas 4 and 1, $T \oplus S$ is a norm-limit of nilpotent operators on $\mathcal{H} \oplus \mathcal{H}$ and $S$ is quasinilpotent. Thus, by a theorem of Rota (cf. [5, p. 77]), $S$ is similar to operators of arbitrarily small norm, and the result follows by another application of Lemma 1.

**Corollary 1.** Suppose $T_1$ and $T_2$ are quasinilpotent operators in $\mathcal{L}(\mathcal{H})$ and both $T_1$ and $T_2^*$ have infinite dimensional kernels. Then $T_1 \oplus T_2$ is the norm-limit of nilpotent operators in $\mathcal{L}(\mathcal{H} \oplus \mathcal{H})$.

**Proof.** Up to unitary equivalence, $T_1 \oplus T_2$ may be written as a $4 \times 4$ matrix
acting on the Hilbert space $\mathcal{H} \oplus \mathcal{H} \oplus \mathcal{H} \oplus \mathcal{H}$, and this matrix, in turn, is unitarily equivalent to

$$
\begin{pmatrix}
0 & B_1 & 0 & 0 \\
0 & C_1 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & B_2 & C_2
\end{pmatrix}.
$$

Since $C_1$ and $C_2$ are quasinilpotent, the corollary now follows from Theorem B and Lemma 3.

We turn now to some preparatory notation that we shall need to review the above-mentioned theorem of Voiculescu. If $T_1$ and $T_2$ are operators in $\mathcal{L}(\mathcal{K}_1)$ and $\mathcal{L}(\mathcal{K}_2)$, respectively, and if there exist a unitary operator $U: \mathcal{K}_1 \to \mathcal{K}_2$ and a compact operator $K$ in $\mathcal{L}(\mathcal{K}_2)$ such that $UT_1U^* = T_2 + K$ and such that $\|K\| < \epsilon$, then we say that $T_1$ and $T_2$ are $\epsilon$-compalent and we write $T_1 \sim T_2(\epsilon)$. Observe that the relation of $\epsilon$-compalence is symmetric, that is, $T_1 \sim T_2(\epsilon)$ if and only if $T_2 \sim T_1(\epsilon)$. We denote the ideal of compact operators in $\mathcal{L}(\mathcal{K})$ by $K(\mathcal{K})$, and the quotient map of $\mathcal{L}(\mathcal{K})$ onto the Calkin algebra $\mathcal{L}(\mathcal{K})/K(\mathcal{K})$ by $\pi$. If $T \in \mathcal{L}(\mathcal{K})$, we denote by $C^*_s(T)$ the separable $C^*$-subalgebra of $\mathcal{L}(\mathcal{K})/K(\mathcal{K})$ generated by $\pi(T)$ and $\pi(1_{\mathcal{K}})$.

**Theorem C** [7]. Let $T \in \mathcal{L}(\mathcal{K})$, and let $\rho$ be a faithful representation of $C^*_s(T)$ on a separable Hilbert space $\mathcal{H}$ such that $\rho(\pi(1_{\mathcal{K}})) = 1_{\mathcal{H}}$. Then for every positive number $\epsilon$, $T \oplus \rho(\pi(T)) \sim T(\epsilon)$.

Based on the above considerations we are now prepared to give a transparent proof of the following theorem.

**Theorem D** [2]. Every quasinilpotent operator $T$ in $\mathcal{L}(\mathcal{K})$ is a norm-limit of nilpotent operators.

**Proof.** Using Theorem C and Lemma 1, we observe that it suffices to construct some faithful representation $\rho$ of $C^*_s(T)$ on a separable Hilbert space $\mathcal{K}$ such that $\rho(\pi(1_{\mathcal{K}})) = 1_{\mathcal{K}}$ and such that $T \oplus \rho(\pi(T))$ is a norm-limit of nilpotent operators in $\mathcal{L}(\mathcal{K} \oplus \mathcal{K})$. Furthermore, if $\tau$ is a faithful representation of $C^*_s(T)$ on a separable Hilbert space $\mathcal{K}$ such that $\tau(\pi(1_{\mathcal{K}})) = 1_{\mathcal{K}}$, then $\mathcal{K}$ may be taken to be $\mathcal{K} \oplus \mathcal{K}$ and $\rho$ to be $\tau \oplus \tau$. Thus it suffices to find a representation $\tau$ with the above-mentioned properties such that $[T \oplus \tau(\pi(T))] \oplus \tau(\pi(T))$ is a norm-limit of nilpotent operators. On the other hand, by virtue of Corollary 1, it suffices to construct such a representation $\tau$ with the property that both $\tau(\pi(T))$ and $\tau(\pi(T^*))$ have infinite dimensional kernels, and this goes as follows. Since $T$ and $T^*$ are quasinilpotent, one
knows from [8] (cf. also [3, Theorem 3.1]) that there exist projections $E$ and $F$ in $\mathcal{L}(\mathcal{H})$ of infinite rank such that $\pi(T)\pi(E) = 0$ and $\pi(T^*)\pi(F) = 0$. Furthermore, $E$ may be written as $E = \sum_{n=1}^{\infty} E_n$, where $\{E_n\}$ is an orthogonal sequence of projections each of which has infinite rank, and similarly $F$ may be written as $F = \sum_{n=1}^{\infty} F_n$. Let $\mathfrak{B}$ be the (separable) $C^*$-subalgebra of $\mathcal{L}(\mathcal{H})/K(\mathcal{H})$ generated by $\pi(T)$, $\pi(1_\mathcal{H})$, and the countable families $\{\pi(E_n)\}_{n=1}^{\infty}$ and $\{\pi(F_n)\}_{n=1}^{\infty}$, and let $\nu$ be a faithful representation of $\mathfrak{B}$ on a separable Hilbert space $\mathcal{H}$ so chosen that $\nu(\pi(1_\mathcal{H})) = 1_\mathcal{H}$. Then $\nu(\pi(T))\nu(\pi(E_n)) = 0$ and $\nu(\pi(T^*))\nu(\pi(F_n)) = 0$ for all positive integers $n$, so clearly the operators $\nu(\pi(T))$ and $\nu(\pi(T^*)) = (\nu(\pi(T)))^*$ have infinite dimensional kernels, and the result follows by setting $\tau = \nu|_{C^*_\varepsilon(T)}$.

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